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FREE BOUNDARIES AND JETS IN THE THEORY OF CAVITATION*

By David Gilbarg and James Serrin

1. Introduction. The problems of finite cavity formation arising in high speed water entry of projectiles and in cavitation phenomena demand an extension of the classical Helmholtz free boundary theory, for which the cavity (or wake) is infinite in extent. We discuss here a theory of the finite cavity based on a model whose characteristic feature is a jet extending from the rear of the cavity into its interior, (Fig. 1). The appropriateness of this model is indicated by water entry and water tunnel experiments which show clearly the existence of jets [1], [11].

Kreisel [2] first considered the theory of these flows, treating the special case of flat plate and wedge obstacles which are symmetrically disposed to the incident uniform flow. Recently, Shiffman [3], using a new approach to the free boundary problem based on the reflection principle, extended the theory of the jet model to include asymmetric flows past polygonal obstacles.

We present here a systematic account of the free boundary theory, showing first the mathematical necessity for the jet model of the cavity; (Sec. 2). The flow problem is solved in explicit form for an inclined flat plate and wedge, (Sec. 4), although the methods apply also to general polygonal obstacles. The considerations here, as in the classical theory, rest on the conformal map of the velocity plane on the plane of the complex velocity potential, and differ therein from Shiffman's method which is independent of the velocity plane. The mappings depend on a generalization of the Schwarz-Christoffel formula which applies to a certain class of polygonal Riemannian manifolds in addition to plane polygons. Finally, in analogy with the Levi-Civita theory of the classical free boundary flows, the flow problem for curved obstacles is reduced to a non-linear boundary value problem for an analytic function defined in the unit circle in terms of which the various flow quantities are expressed by simple formulas, (Sec. 5).

Attention should be called to a note by Efros [6] which recently became available to the authors, where some of the results of the present paper are stated without derivation.

2. Preliminary considerations. We consider a uniform two dimensional flow of an ideal incompressible fluid striking a fixed obstacle. It is assumed that two free streamlines separate from the obstacle and enclose a region of constant pressure. This region may be thought of as the wake of the classical Helmholtz theory, or the cavity associated with water entry and water tunnel cavitation phenomena.

Flows satisfying the following conditions will be considered:

(i) the flow is steady and irrotational;

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(ii) the pressure in the interior of the fluid is nowhere less than the cavity pressure, p_c;

(iii) the static pressure of the uniform flow is greater than the cavity pressure; i.e., $p_\infty > p_c$ where p_∞ is the pressure at infinity;

(iv) the pressure is continuous across the free streamlines.

Condition (ii) reflects the fact that cavitation occurs in a real fluid when the pressure is less than the vapor pressure of the fluid. It should be noted that these conditions differ only in (iii) from the classical theory, for which $p_c = p_\infty$.

The constant flow speed, V, on the free streamlines may be determined from Bernoulli's equation,

$$p + 1/2\,\rho q^2 = p_\infty + 1/2\,\rho U^2;$$

where p is the pressure, ρ the constant density, and q the flow speed, at any point in the flow, and U is the velocity at infinity. Thus V is related to the other constants by

$$p_c + 1/2\,\rho V^2 = p_\infty + 1/2\,\rho U^2.$$

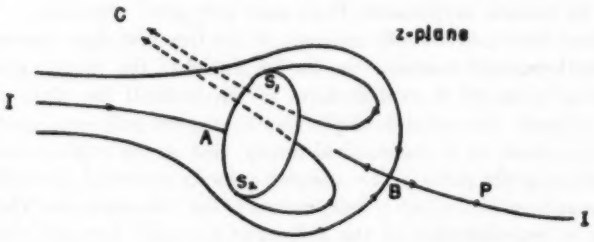

FIG. 1

From (ii) we have that the pressure minimum, and, as a result, the velocity maximum are achieved on the free boundary. It follows readily that the free streamlines are convex towards the cavity.

The behavior of the free streamlines will indicate the direction of further considerations. Several possibilities are available.

(a) The free streamlines meet at a point, forming a finite cavity bubble behind the obstacle. This is impossible, since it would violate obvious continuity conditions at the junction of the two streamlines.

(b) The streamlines extend to infinity, as in the classical theory. A consideration of the conformal mapping of the flow region on its image in the velocity plane shows this to be impossible also[1].

[1] If $w(z)$ is the complex velocity potential of the flow, the image region in the velocity plane, obtained from the mapping $q(z) = dw/dz$, is a simply connected region. As a consequence of the assumption that the free streamlines extend to infinity, the point at infinity, I, in the flow plane must map into a set of points, instead of a single point, on the boundary of this image region. But, according to the theory of the correspondence of boundaries under conformal mapping [7], this is the case only if the image points of I, (which form a prime end), are inaccessible from the interior of the image region. One can see from the simple nature of the remainder of the boundary in the q-plane that this is not possible.

(c) The streamlines turn back, forming a jet. (See Fig. 1.) Such a jet would strike the rear of the obstacle or the free streamlines, resulting in an unsteady flow. If, however, the non-physical assumption is made that the jet is not interrupted, or more precisely, that the jet forms a second sheet of the flow plane, the flow is still mathematically possible.

Since (c) is the only mathematically tenable alternative consistent with classical hydrodynamics[2], this model of cavity flow would seem to be the "correct" one. Furthermore, water tunnel and water entry experiments [1], [11] show the presence of such backward jets. These flows are the concern of the present article.

It may be shown, as in (b), that the jet is of finite width, and it is clear that the formation of the jet requires the flow to contain a stagnation point in its interior as well as on the obstacle (provided it is non-cusped). The only type of stagnation point consistent with a uniform flow at infinity and two free streamlines is one with two entering and two outgoing streamlines, and it is supposed that there is exactly one such point. The essential features of the flow are displayed in Fig. 1.

3. Statement of the problem. Let the region of flow be in the z-plane, where $z = x + iy$, the x-axis being taken parallel to the incident uniform flow. Let

$$w(z) = \varphi(x, y) + i\psi(x, y),$$

where φ is the velocity potential, ψ is the stream function, and $w(z)$ is the complex velocity potential defined over the two sheeted flow region R in the z-plane. The flow is determined once $w(z)$ and the free streamlines, Σ_1, Σ_2, are known. These will ultimately be obtained through the knowledge of the local properties of $w(z)$, which are completely stated in the following.

(i) In the neighborhood of the stagnation point B, $w(z)$ is of the form,

$$w(z) = w_B + (z - z_B)^2 P(z - z_B), \qquad P(0) \neq 0.^3$$

(ii) In the neighborhood of infinity, I, $w(z)$ is of the form

$$w(z) = \kappa \log z + Uz + P(1/z), \qquad \kappa \neq 0.$$

For consider any circuit around the cavity in the z-plane. The value of $w(z)$ after completing such a circuit in, say, the positive direction, is seen to differ from its starting value by a constant $k \neq 0$ which is independent of the path and the starting point. In particular, the change in value of the stream function, ψ, must be non-zero to account for the jet, while the change in value of the

[2] Other variations of (c) are still possible, e.g., the jet might strike another portion of the obstacle, dividing there into two jets which form additional sheets of the flow plane. However, these possibilities will be disregarded here.

[3] The values in the various complex planes corresponding to the points lableled $A, B, C, \cdots$ in the flow plane are designated throughout by the appropriate subscripts; thus the value of w at the point in the w-plane corresponding to B in the low plane is w_B. The notation $P(z - z_0)$ will be used to represent any functions regular at the point z_0, but if a specific function is meant it will be represented differently.

potential, φ, is equal to the circulation of the flow about the cavity. This may or may not vanish. Let $\kappa = k/2\pi i$, so that $w(z) - \kappa \log z$ is single valued in the neighborhood of I. It therefore has a Laurent expansion about I, which, since $\lim_{z \to I} w'(z) = U$, is of the form $Uz + P(1/z)$.

(iii) At all other interior points of R, $w(z)$ is regular and $w'(z) \neq 0$.

The problem can now be formulated in function-theoretic terms without reference to the flow.

Let R be a simply connected region without winding points on a two-sheeted Riemann surface over the z-plane. R is to contain the point at infinity as an interior point, I, and as a boundary point, C. Part of the boundary of R is to be an arc S_1AS_2 of a given curve in the z-plane (the obstacle), while the remainder is to consist of two (unknown) curves Σ_1, Σ_2. These curves are to have continuously turning tangents with the same limiting direction γ at C.

The complex velocity potential is to be an analytic function $w(z)$ defined throughout R which satisfies the following conditions.

1. $|\, dw/dz \,| = V$ on Σ_1, Σ_2.

   $|\, dw/dz \,| \leq V$ on S_1AS_2.[4]

2. The image of the boundary of R is the slit positive real axis of the w-plane.
3. In the neighborhood of some interior point z_B,

$$w(z) = w_B + (z - z_B)^2 P(z - z_B), \qquad P(0) \neq 0.$$

4. In the neighborhood of I,

$$w(z) = \kappa \log z + Uz + P(1/z), \qquad \kappa \neq 0.$$

5. Elsewhere in the interior of R, $w(z)$ is regular and $w'(z) \neq 0$.

A function $w(z)$ and curves Σ_1, Σ_2 which satisfy these conditions form jointly a solution of the free boundary problem for the given obstacle. The constants z_A, z_B, z_{S_1}, z_{S_2}, κ, and γ, which are in general unknown, are to be determined in conjunction with $w(z)$. As we shall see, unless the circulation, $(= -Im(2\pi\kappa))$, or some other real quanity is fixed, the solution will not be unique.

The foregoing is not intended to be a minimum formulation of the flow problem, but does indicate its essential content. The general method of solution, as in the classical treatment, is to reduce the problem with free boundaries to one of conformal mapping of regions with *known* boundaries. The regions concerned are the w- and Ω-images of the flow plane, where

$$\Omega(z) = i \log V^{-1} dw/dz = \theta + i \log q/V,$$

θ being the inclination of the velocity vector.

A description of the w-image is readily obtained from the above properties of $w(z)$. By virtue of 2 and 4 the mapping is infinitely many-valued and the w-plane

[4] This is equivalent to (ii) of Sec. 2. Although an important physical condition, it may be omitted for the sake of generalizing the mathematical problem.

is multiply covered. A single sheet of the Riemann surface over the w-plane (Fig. 2) is obtained by cutting the z-plane along the streamline IBC which joins the branch point I to the boundary. The w-image of the flow plane is then a Riemannian manifold with boundary AS_1CS_2A. The slits IBC in the upper and lower parts of the w-plane are respectively images of the upper and lower sides of the cut IBC in the z-plane, and points on corresponding edges of each

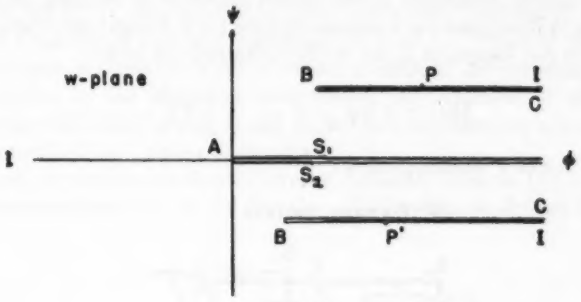

FIG. 2

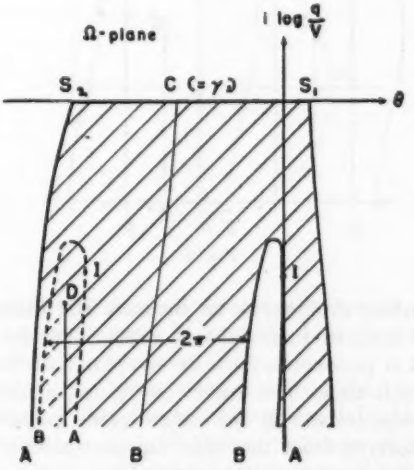

FIG. 3

slit at equal distances from w_B (e.g. P, P'), being images of the same point, are to be identified on the manifold. Considered as a Riemann surface, the complete w-image consists of infinitely many such congruent sheets joined pairwise at the winding points, w_B, along BI and BC.

Consider now the Ω-image of the flow plane, (Fig. 3). If the flow region is cut along the streamline VIA, then $\Omega(z)$ becomes single valued and maps R into a region bounded by a curve AS_1CS_2A, the image of the free boundary, S_1CS_2,

being a segment of the real axis. The two sides of the cut map into two curves BIA, respective points of which are identified if their values differ by 2π. (This is a consequence of the logarithmic singularity of $\Omega(z)$ at z_B.) The Ω-image is thus a Riemannian manifold. If the obstacle S_1AS_2 is a straight line the boundary curve AS_1CS_2A of the manifold is an infinite rectangle. (Fig. 4).

Considered as a Riemann surface, the Ω-image of the uncut z-plane consists of infinitely many congruent sheets joined pairwise at winding points, D, of type z^{2}[5]. Fig. 4 illustrates the Riemann surface for a straight line obstacle. These points, D, are not images of I, for in the neighborhood of I,

$$\Omega(z) = i \log \frac{U}{V} + \frac{i\kappa}{Uz} + \frac{1}{z^2} P(\frac{1}{z}).$$

Hence $\Omega(z_I)$ is not a winding point.

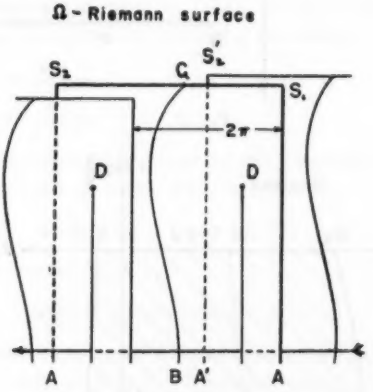

FIG. 4

4. The Free Boundary Problem for an Inclined Flat Plate. If the obstacle is a flat plate, and S_1AS_2 therefore straight, an explicit functional relation between Ω and w is obtained in parametric form by mapping both image regions on the same half plane or unit circle. The explicit mappings, involving here polygonal Riemannian manifolds, instead of the simple plane polygons of the classical theory, are easily derived from the following generalization of the Schwarz-Christoffel formula for the mapping of polygons.

Mapping theorem: Let $u(t)$ be an analytic function which maps the upper half t-plane on a Riemannian manifold with closed polygonal boundary. In the neighborhood of interior points $t = \sigma_j$, $(j = 1, \cdots, n)$, let

$$u'(t) = (t - \sigma_j)^{\alpha_j} P(t - \sigma_j), \qquad P(0) \neq 0,$$

[5] The Ω-image of a simple closed curve in R which is arbitrarily close to the path AS_1CBCS_2A (with BC exterior to the curve) is seen to be a curve winding twice about the points in the region $AS_1S_2'A'A$ (Fig. 4). It can be proved that this region therefore contains exactly one winding point, D, which is two-sheeted.

where α_j is real and unequal to zero. At all other interior points let $u(t)$ be regular and $u'(t) \neq 0$. If the boundary has interior angles $\beta_k\pi$, $(k = 1, \cdots, m)$, at vertices corresponding to $t = \tau_k(\tau_k$ real), then $u(t)$ is of the form

$$(1) \quad u(t) = A \int^t \{\prod_j [t^2 - (\sigma_j + \bar{\sigma}_j)t + |\sigma_j|^2]^{\alpha_j} \prod_k (t - \tau_k)^{\beta_k - 1}\}\, dt + B$$

where A, B are complex constants. If the manifold is given, then the precise values of A, B, σ_j, τ_k, are determined from the known positions of the vertices, $u(\tau_k)$, and from the given data concerning $u(t)$ at σ_j. Three of the constants τ_k can be chosen arbitrarily; if one is chosen at infinity the corresponding factor will not appear in the integrand. Conversely, any function of the form (1) has the prescribed local behavior and maps the half plane on a polygonal Riemannian manifold with vertex angles $\beta_k\pi$.

The proof, as for the classical Schwarz-Christoffel formula ([7] pp. 423–426), follows from consideration of the function u''/u', which is defined by reflection

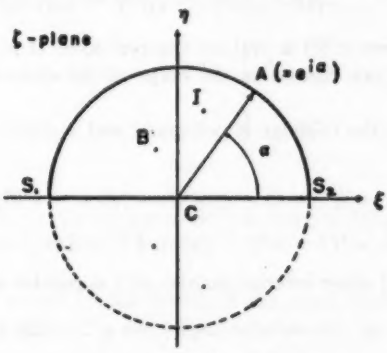

FIG. 5

in the entire plane, and is regular except for simple poles of strength, α_j, α_j, $\beta_k - 1$, at σ_j, $\bar{\sigma}_j$, τ_k, respectively, from which (1) is obtained readily. We omit a fuller discussion here. (For details see [10]).

It will be convenient for formal reasons to map the image regions into the upper half unit circle, Γ, of the ζ plane; $w(z)$ can be obtained through the resulting parametric representations. By well-known theorems on conformal mapping, there is a unique analytic function $z(\zeta)$ which maps Γ conformally on R such that $z(0) = C$, $z(-1) = S_1$, $z(+1) = S_2$; (Fig. 5). The fact that the mapping is conformal gives the expansion at ζ_1,

$$(2) \qquad z(\zeta) = r/(\zeta - \zeta_1) + P(\zeta - \zeta_1), \qquad (r \neq 0),$$

while at all other points in Γ, $z'(\zeta) \neq 0$. The functions $w(\zeta) \equiv w(z(\zeta))$, $\omega(\zeta) \equiv \Omega(z(\zeta))$ are then the required mappings of Γ on the image regions.

An obvious modification of the mapping theorem enables us to find $w(\zeta)$ and $\omega(\zeta)$. Namely, Γ is first mapped on the upper half plane by $t = -\frac{1}{2}(\zeta + \zeta^{-1})$ and

then onto the polygonal image regions by means of the mapping theorem; this substitution replaces any linear factor $(t - t_0)$ in (1) by $\dfrac{-(\zeta - \zeta_0)(\zeta - 1/\zeta_0)}{2\zeta}$, where $t_0 = -\frac{1}{2}(\zeta_0 + 1/\zeta_0)$, and introduces the factor $dt/d\zeta = -\frac{1}{2}(\zeta^2 - 1)/\zeta^2$ in the integrand. From the prescribed local behavior of $w(z)$ we have,

$$
\begin{aligned}
&\text{at } \zeta_B,\ w'(\zeta) = (\zeta - \zeta_B)P(\zeta - \zeta_B),\ P(0) \neq 0;\\
&\text{at } \zeta_I,\ w'(\zeta) = \frac{-U\tau}{(\zeta - \zeta_I)^2} - \frac{\kappa}{(\zeta - \zeta_I)} + P(\zeta - \zeta_I);\\
&\text{at all other interior points, } w(\zeta) \text{ is regular and } w'(\zeta) \neq 0.
\end{aligned}
\tag{3}
$$

At the vertices, A, C, of the boundary, the interior angles are 2π and 0. Hence, making the substitution, $t = -\frac{1}{2}(\zeta + 1/\zeta)$, in (1), we have,

$$
\frac{dw}{d\zeta} = M\,\frac{(1-\zeta^2)(\zeta - e^{i\alpha})(\zeta - e^{-i\alpha})(\zeta - \zeta_B)(\zeta - \bar{\zeta}_B)(\zeta - 1/\zeta_B)(\zeta - 1/\bar{\zeta}_B)}{\zeta(\zeta - \zeta_I)^2(\zeta - \bar{\zeta}_I)^2(\zeta - 1/\zeta_I)^2(\zeta - 1/\bar{\zeta}_I)^2},
\tag{4}
$$

where M is real since $w'(\zeta)$ is real on the real axis. It is to be noted that the form of $dw/d\zeta$ does not depend on the shape of the obstacle. $w(\zeta)$ itself is never explicitly needed.

For the flat plate the Ω-image is polygonal and a similar process gives $d\omega/d\zeta$. That is,

$$
\begin{aligned}
&\text{at } \zeta_D,\ \omega'(\zeta) = (\zeta - \zeta_D)P(\zeta - \zeta_D)\ ; \qquad P(0) \neq 0\ ;\\
&\text{at } \zeta_B,\ \omega'(\zeta) = i/(\zeta - \zeta_B) + P(\zeta - \zeta_B);\\
&\text{at all other interior points, } \omega(\zeta) \text{ is regular and } \omega'(\zeta) \neq 0.
\end{aligned}
$$

At the vertices S_1, S_2, the interior angles are $\pi/2$, while at A the interior angle is 0. Hence,

$$
\frac{d\omega}{d\zeta} = K\,\frac{(\zeta - \zeta_D)(\zeta - \bar{\zeta}_D)(\zeta - 1/\zeta_D)(\zeta - 1/\bar{\zeta}_D)}{(\zeta - e^{i\alpha})(\zeta - e^{-i\alpha})(\zeta - \zeta_B)(\zeta - \bar{\zeta}_B)(\zeta - 1/\zeta_B)(\zeta - 1/\bar{\zeta}_B)}.
$$

Separating into partial fractions, and noting that the residue at $\zeta = \zeta_B$ is i, and therefore is $-i$, $-i$, $+i$ at $\zeta = \bar{\zeta}_B$, $1/\zeta_B$, $1/\bar{\zeta}_B$, respectively, while

$$
\operatorname{Re}[\omega(e^{i\sigma})]\,\Big|_{\sigma=\alpha+0}^{\sigma=\alpha-0} = \pi,
$$

we have, after integrating,

$$
\omega(\zeta) = i \log L\left(\frac{\zeta - e^{i\alpha}}{\zeta - e^{-i\alpha}}\right)\left(\frac{\zeta - \zeta_B}{\zeta - \bar{\zeta}_B}\right)\left(\frac{\zeta - 1/\bar{\zeta}_B}{\zeta - 1/\zeta_B}\right).
\tag{5}
$$

This formula can be derived more simply without use of the mapping theorem. From the nature of the singularities of $\omega(\zeta)$ at $\zeta = \zeta_B$ and $\zeta = e^{i\alpha}$, we infer

$$
\omega(\zeta) = i \log \frac{(\zeta - e^{i\alpha})(\zeta - \zeta_B)}{(\zeta - e^{-i\alpha})(\zeta - \bar{\zeta}_B)} + f(\zeta),
$$

$f(\zeta)$ being regular in the entire unit circle, $|\zeta| \leq 1$, in which $\omega(\zeta)$ is defined by virtue of the Schwarz reflection principle. Since $\omega(\zeta)$ has a constant real part on $|\zeta| = 1$, we may reflect out of the unit circle to obtain $\omega(\zeta)$ in the entire plane in the form,

$$\omega(\zeta) = i \log \frac{(\zeta - e^{i\alpha})(\zeta - \zeta_B)(\zeta - 1/\bar{\zeta}_B)}{(\zeta - e^{-i\alpha})(\zeta - \bar{\zeta}_B)(\zeta - 1/\zeta_B)} + g(\zeta),$$

where $g(\zeta)$ is regular in the entire plane and therefore a constant, from which (5) follows. This method could also be used to obtain (4).

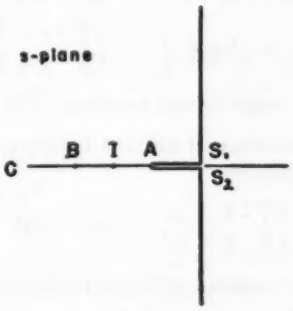

FIG. 6

The constants L, M, ζ_I, ζ_B, α, are subject to the conditions,

1. $\omega(\zeta)$ real for ζ real, so that $|L| = 1$;
2. $\omega(\zeta_I) = i \log U/V$;
3. $z_{S_1} - z_{S_2} = le^{i\delta}$, where l is the length of the plate, and δ its inclination;
4. M is real;
5. $$\lim_{\zeta \to \zeta_I} \frac{d}{d\zeta}\left[(\zeta - \zeta_I)^2 e^{i\omega(\zeta)} \frac{dw}{d\zeta}\right] = 0.$$

This states that the residue of $dz/d\zeta$ at ζ_I is 0, which follows from (2).

If, furthermore, the circulation is prescribed, say equal to H, then by (3),

$$2\pi \operatorname{Im}\left\{\lim_{\zeta \to \zeta_I} \frac{d}{d\zeta}\left[(\zeta - \zeta_I)^2 \frac{dw}{d\zeta}\right]\right\} = -2\pi \operatorname{Im}(\kappa) = H.$$

There are now nine real conditions to determine the nine real constants contained in L, M, ζ_I, ζ_B, α. Conditions 1.–3. and the mapping formulas insure that $w(\zeta)$ and $\omega(\zeta)$ define a function $w(z)$ which satisfies the requirements of Sec. 3. The free boundaries are determined as the image of the real ζ axis. Further details concerning the flow will be deduced later.

Similar arguments apply to the flow problem for a wedge obstacle, provided the stagnation point A occurs at the vertex. If the wedge angle is $\epsilon\pi$, it is easy to see that the function $\omega(\zeta)$ is of the form

$$\omega(\zeta) = i \log L\left(\frac{\zeta - e^{i\alpha}}{\zeta - e^{-i\alpha}}\right)^{\epsilon} \frac{(\zeta - \zeta_B)(\zeta - 1/\bar{\zeta}_B)}{(\zeta - \bar{\zeta}_B)(\zeta - 1/\zeta_B)}. \tag{6}$$

Since this formula requires that the stagnation point be at the vertex, the circulation can no longer be fixed beforehand. The methods used in this section are in principle applicable also to arbitrary polygonal obstacles, where, in general, a free streamline separates from each vertex in contact with the flow.

As an application of the above, consider the symmetric flow past a vertical flat plate. The change of variables $s = \frac{1}{2}i(\zeta - \zeta^{-1})$ effects a simplification in (4) and (5) leading to the formulas

$$\frac{dw}{ds} = \frac{dw}{d\zeta}\frac{d\zeta}{ds} = M_1 \frac{s(s^2 - b^2)}{(s^2 - h^2)^2}$$

$$\omega(s) = i \log L_1 \left[\frac{s+1}{s-1}\right]^{1/2} \frac{s-b}{s+b},$$

where $b = s_B$, $h = s_I$, are negative real numbers; (Fig. 6). Since $\lim_{s \to \infty} \omega(s) = -\pi$ we have $L_1 = -1$. From conditions 2 and 5 of the present section we obtain after a brief calculation the equations

$$\frac{U}{V} = \left[\frac{h+1}{h-1}\right]^{1/2} \frac{b-h}{b+h}, \qquad b = -2h^2 - h + 2,$$

which determine b and h as functions of the cavitation number N, where

$$N = \frac{V^2}{U^2} - 1 = \frac{-4h^3 + h^2 + 4h - 2}{(h^2 + h - 1)^2}.$$

An estimate of the drag may be made using the formula

$$\text{Drag} = \int_{-l/2}^{+l/2} (p - p_c)\, dy,$$

$$= \frac{\rho l V^2}{2}\left[1 - \frac{\int_{-1}^{0}\left[\frac{1+s}{1-s}\right]^{1/2}\left[\frac{s-b}{s^2-h^2}\right]^2 s\,ds}{\int_{-1}^{0}\left[\frac{1-s}{1+s}\right]^{1/2}\left[\frac{s+b}{s^2-h^2}\right]^2 s\,ds}\right].$$

For $0 \leq N \leq 1$ the quantities $(s-b)/(s^2-h^2)$ and $(s+b)/(s^2-h^2)$ are essentially constant $(=b/h^2)$ over the interval of integration. Hence in this physically important range of N values the drag is easily approximated (within an accuracy of 2 per cent) by

$$\frac{2\pi}{4+\pi}\,\frac{\rho l U^2}{2}\,(1+N). \tag{7}$$

This formula was first stated by Zoller [8] on the basis of a different model of the cavity, and is in agreement with experiments. When $N = 0$, which is the classical case, the above expression is exact and we derive the classical formula for the drag, $[\pi/(4+\pi)]\rho l U^2$.

5. Free boundary problem for curved obstacles. Direct conformal mapping procedures, as in the preceding section, are no longer adequate to handle the

problem of curved obstacles, since for these the Ω-plane is not explicitly known. However, in analogy with Levi-Civita [4, 5], the flow problem can be formulated as a non-linear boundary value problem for the unknown function, $\omega(\zeta)$. We see that $\omega(\zeta) = \theta + i\tau$, $(\tau = \log q/V)$, must satisfy the following conditions in the semi circle, Γ.

1. $\tau = 0$ for ζ real;
2. $\tau \leq 0$ for $|\zeta| = 1$; (see footnote 4);
3. in the interior of Γ, $\omega(\zeta)$ has a singularity at ζ_B, where

$$\omega(\zeta) = i \log (\zeta - \zeta_B) + P(\zeta - \zeta_B),$$

and is regular elsewhere;

(8) 4. $$\lim_{\zeta \to \zeta_1} \frac{d}{d\zeta}\left[(\zeta - \zeta_1)^2 e^{i\omega(\zeta)} \frac{dw}{d\zeta}\right] = 0,$$

where $dw/d\zeta$ is given by (4),

5. $$\omega(\zeta_1) = i \log U/V$$

6. $$z(\sigma) = \frac{1}{V}\int_{e^{i\alpha}}^{e^{i\sigma}} e^{i\omega(\zeta)} \frac{dw}{d\zeta}\, d\zeta, \qquad (0 \leq \sigma \leq \pi),$$

is to be a parametric representation for (a portion of) the fixed boundary, (the origin thereby being fixed at the stagnation point).

It is clear, conversely, that any function $\omega(\zeta)$ satisfying conditions 1–5 defines a flow with free boundaries. In addition, 6 insures that the obstacle is a streamline of the flow, $\omega(\zeta)$ therefore solving the flow problem for the given fixed boundary. The proof of existence or uniqueness of $\omega(\zeta)$ for a flow against a given obstacle with prescribed U, V is as yet an open problem.

As in the Levi-Civita theory, the various flow quantities of interest are expressible in simple form in terms of $\omega(\zeta)$. Thus,

$$z(\xi) = \frac{1}{V}\int_{e^{i\alpha}}^{\xi} e^{i\omega(\zeta)} \frac{dw}{d\zeta}\, d\zeta, \qquad \xi \text{ real}, -1 \leq \xi \leq +1,$$

defines the free streamlines in parametric form, $\xi = \pm 1$ defining the separation poinst, S_1, S_2. The jet width, J, and limiting direction, γ, are given by

(9) $$J = -2\pi \operatorname{Re}(\kappa)/V \qquad \gamma = \omega(0).$$

J is determined also through the relation

(10) $$-2iJe^{i\gamma} = \frac{1}{V}\oint e^{i\omega(\zeta)} \frac{dw}{d\zeta}\, d\zeta,$$

where the circuit is about the origin. The integral is easily evaluated by residues from (4) and (9), yielding

$$J = -\pi M/V.$$

If X is the drag of the obstacle, Y the lift, then the force, (assuming the unwetted portion of the obstacle is subject to the cavity pressure, p_c), is given by

$$F = X + iY = -\frac{i\rho V}{2}\oint e^{i\omega(\zeta)} \frac{dw}{d\zeta}\, d\zeta$$

the integral being taken around $|\zeta| = 1$. The standard derivation of this formula [5] applies here without essential modification. The residues of the integrand for $|\zeta| < 1$ are non-zero only at $\zeta = 0$ and $\zeta = \bar{\zeta}_1$, as one ascertains from (4) and (8). At the origin and $\bar{\zeta}_1$ the residues are, respectively, $\frac{1}{2}i\rho VMe^{i\gamma}$ and $(U/V)[N + iP\omega'(\bar{\zeta}_1)]$, where

$$N = \lim_{\zeta \to \bar{\zeta}_1} \frac{d}{d\zeta}\left[(\zeta - \zeta_1)^2 \frac{dw}{d\zeta}\right] = -\bar{\kappa}$$

$$P = \lim_{\zeta \to \bar{\zeta}_1} \left[(\zeta - \zeta_1)^2 \frac{dw}{d\zeta}\right] = -U\bar{\tau}.$$

Since $\omega'(\bar{\zeta}_1) = -i\bar{\kappa}/U\bar{\tau}$, we obtain after simplifying,

$$F = -\rho(V^2 J e^{i\gamma} + 2\pi U\bar{\kappa}),$$

so that

$$X = \rho VJ(U - V\cos\gamma) \quad , \quad Y = -\rho(U^2 J \sin\gamma + UH). \tag{11}$$

These formulas can also be obtained from momentum considerations (see Birkhoff [9] where the derivation is given for symmetric flows).

An approximate value of the jet width for the symmetric flow past a vertical flat plate is obtained by comparing (11) for $\gamma = -\pi$ and (7); thus

$$J = \frac{l\pi}{4 + \pi}\left(\frac{V}{U + V}\right).$$

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SYNTHESIS OF A FINITE $2n$-TERMINAL NETWORK BY A GROUP OF NETWORKS EACH OF WHICH CONTAINS ONLY ONE OHMIC RESISTANCE

By Yosiro Oono

1. Introduction. With the progress of electrical communication engineering, the theory of electric network synthesis has made a great advancement. But its fundamental problem, namely the problem of $2n$-terminal network synthesis from prescribed characteristic functions has remained unsolved, since C. M. Gewertz solved the case of four-terminal networks.[1]

This paper solves this problem and presents a general method of synthesizing a finite passive $2n$-terminal network from its prescribed impedance matrix.[2] In the theory of electric wave filters, it is the well-known fact that every impedance function is given by the driving-point impedance of a reactive four-terminal network terminated in a resistance at its one end.[3] In this paper, we have a general conclusion that every $2n$-terminal network can be represented by a reactive $4n$-terminal network terminated in n resistances at its n pairs of terminals. Also a basic property of reactive networks is made clear by extending the fact that a reactive four-terminal network is characterized by two functions.[4]

Thus, although the present theory involves Gewertz's theory as a very particular case, it may rather be regarded as the extension of the theory of electric wave filters.

2. Notations and definitions[5]

λ the complex variable.

j $\sqrt{-1}$.

ω the angular frequency.

$Re\, f(\lambda)$ the real part of $f(\lambda)$.

$$Re\, F(\lambda) = \| Re\, f_{st}(\lambda) \| = \begin{Vmatrix} Re\, f_{11}(\lambda) & \cdots & Re\, f_{1n}(\lambda) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ Re\, f_{n1}(\lambda) & \cdots & Re\, f_{nn}(\lambda) \end{Vmatrix}$$

called the real part matrix of a matrix $F(\lambda) = \| f_{st}(\lambda) \|$.

w_{st} $(s, t = 1, 2, \cdots, n)$

open-circuit driving-point ($s = t$) or transfer ($s \neq t$) impedances of a $2n$-terminal network. (By the term "network" we mean the finite passive network with lumped linear elements.) The

[1] C. M. Gewertz, Network Synthesis (Baltimore, 1933).

[2] As to the definitions see paragraph 2.

[3] S. Darlington, J. of Math. & Phys. **18**, 257 (1939). W. Cauer, T. F. T. **29**, 185 (1940).

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[5] W. Cauer, Theorie der linearen Wechselstromschaltungen Bd. 1 (Berlin, 1941). This book gives a complete preliminary introduction to the subject of this paper.

$w_{st} : s$ are defined by the relations between voltages and currents at n pairs of terminals (Fig. 1), i.e. by a matrix equation

$$\begin{Vmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_n \end{Vmatrix} = \begin{Vmatrix} w_{11}\, w_{12} \cdots w_{1n} \\ w_{21}\, w_{22} \cdots w_{2n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ w_{n1}\, w_{n2} \cdots w_{nn} \end{Vmatrix} \begin{Vmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ I_n \end{Vmatrix}.$$

$W = \| w_{st} \|$ the impedance matrix.

$Z = \| z_{st} \|$ used in place of W, when the network is purely reactive.

Network representation

If a matrix is the impedance matrix of a physically realizable network, it is said to have a network representation.[6]

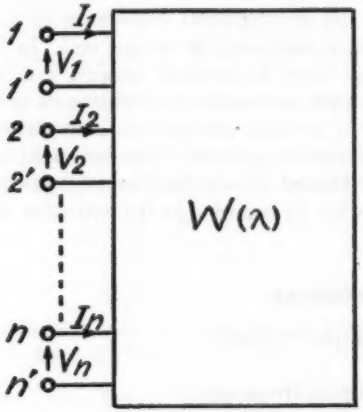

FIG. 1

Positive real function

a function $f(\lambda)$ which satisfies the following conditions:

(1) $f(\lambda)$ is real for λ real.

(2) $f(\lambda)$ is analytic for $Re\ \lambda > 0$.

(3) $Re\, f(\lambda) \geqq 0$ for $Re\ \lambda > 0$.

A function which satisfies (1) is called a real function.

Positive definite matrix

the matrix of a positive definite quadratic form.[7]

Positive real matrix[8]

a matrix $F(\lambda) = \| f_{st}(\lambda) \|$ whose quadratic form

$$\sum_{t=1}^{n} \sum_{s=1}^{n} f_{st}(\lambda) x_s x_t$$

[6] O. Brune, J. of Math. & Phys. 10, 191 (1931).

[7] Including the so-called semi-definite form.

[8] Reference 1, p. 84. Reference 5, p. 127.

is a positive real function for all real values of the x:s. It is the established fact that the impedance matrix must be a positive real matrix.[9] (Elements of the positive real matrix are assumed to be rational functions throughout this paper, since networks under consideration are the finite ones.)

Hurwitz polynomial

a polynomial with real coefficients having no zeros in the right-half plane and on the boundary.

3. Method of synthesis. In the following we shall concern ourselves with impedance matrices, since similar statements hold for admittance matrices.

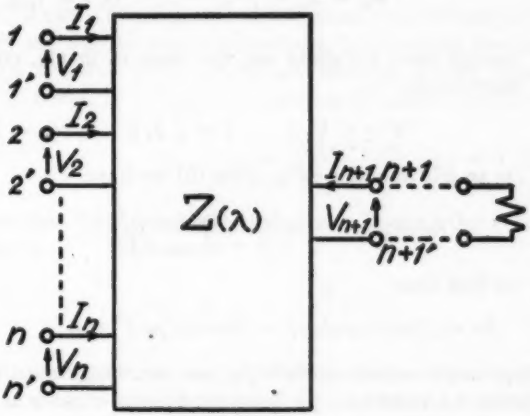

FIG. 2

Suppose we have a reactive $2(n+1)$-terminal network shown in Fig. 2. Then we have a matrix equation

$$\begin{Vmatrix} V_1 \\ \cdot \\ \cdot \\ V_{n+1} \end{Vmatrix} = \begin{Vmatrix} z_{11} & \cdots & z_{1,n+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ z_{n+1,1} & \cdots & z_{n+1,n+1} \end{Vmatrix} \begin{Vmatrix} I_1 \\ \cdot \\ \cdot \\ I_{n+1} \end{Vmatrix}$$

or

$$V = ZI \tag{1}$$

where

$$Z = \| z_{st} \|, \qquad V = \| V_s \|, \qquad I = \| I_s \| \tag{2}$$

$$(s, t = 1, 2, \cdots, n+1).$$

[9] Reference 1, p. 53. Reference 5, p. 136. E. A. Guillemin, Communication Networks Vol. 2 (New York, 1935), p. 217.

Now if we terminate this network in a resistance (being assumed to be unity without any loss of generality) at end $(n+1)$ (Fig. 2) and eliminate V_{n+1} and I_{n+1} from (1) noting that

$$V_{n+1} = -1 \cdot I_{n+1},$$

we shall readily get

$$V = WI, \tag{3}$$

where

$$W = \| w_{st} \|, \tag{4}$$

$$w_{st} = z_{st} - \frac{z_{s,n+1} z_{t,n+1}}{1 + z_{n+1,n+1}} \quad (s, t = 1, 2, \cdots, n) \tag{5}$$

and V and I, though their notations are the same as in (1), do not contain V_{n+1} and I_{n+1} respectively, i.e.

$$V = \| V_s \|, \qquad I = \| I_s \| \qquad (s = 1, 2, \cdots, n).$$

Since every z_{st} is an odd function of λ, from (5) we have

$$Re\ w_{st}(j\omega) = -\frac{z_{s,n+1}(j\omega) z_{t,n+1}(j\omega)}{1 - z^2_{n+1,n+1}(j\omega)}. \tag{6}$$

By using this we find that

$$Re\ w_{ss}(j\omega) Re\ w_{tt}(j\omega) - \{Re\ w_{st}(j\omega)\}^2 \equiv 0 \tag{7}$$

i.e. all 2-rowed principal minors of $ReW(j\omega)$ are identically zero. Furthermore, by easy calculation we know that all 3-rowed principal minors of $ReW(j\omega)$ are also identically zero. (This can be known without calculation as the natural result of (7), because $ReW(j\omega)$ is a positive definite matrix for any value of ω.) Therefore the rank [10] of $ReW(j\omega)$ must be 1 or 0.[11] If we exclude the case of rank 0, which means that $Rew_{st}(j\omega) \equiv 0$ $(s,t = 1, 2, \ldots, n)$, i.e. the end $(n+1)$ is independent of others, we may have:

THEOREM 1. If a $2n$-terminal network is made by terminating a reactive $2(n+1)$-terminal network in a resistance at one pair of terminals, the real part matrix of its impedance matrix is of rank 1 along the whole axis of imaginaries.

Next we shall prove the converse theorem stated as follows.

THEOREM. 2. A positive real matrix of order n, whose real part matrix is of rank 1 on the imaginary axis, is capable of representation by a reactive $2(n+1)$-terminal network terminated in a resistance at its one pair of terminals.

[10] We say $ReW(j\omega)$ to be of rank r when not all its minor determinants of order r identically vanish, while all of order $r+1$ do so.

[11] If all the $(r+1)$-rowed principal minors of the symmetrical matrix A are zero, and also all the $(r+2)$-rowed principal minors, then the rank of A is r or less.

Let

$$W = \| w_{st} \| \qquad (s, t = 1, 2, \cdots, n)$$

be the given matrix. If w_{st} has poles on the imaginary axis, whose principal parts are summed up in w'_{st}, we may separate

$$W' = \| w'_{st} \|$$

from W. W' is a positive real matrix[12] which can be realized into a reactive $2n$-terminal network by the method due to W. Cauer.[13] If we write

$$W = W' + W'', \tag{8}$$

$W'' = \| w''_{st} \|$ is also a positive real matrix whose real part matrix is of rank 1 on the imaginary axis and w''_{st} has no poles there. Let us study the problem whether there exists a reactive $2(n+1)$-terminal network whose impedance matrix $Z = \| z_{st} \|$ $(s, t = 1, 2, \cdots, n+1)$ satisfies the similar relations to (5)

$$w''_{st} = z_{st} - \frac{z_{s,n+1} z_{t,n+1}}{1 + z_{n+1,n+1}} \quad (s, t = 1, 2, \cdots, n). \tag{9}$$

First, in order that the driving-point impedance at end s (for example) may give w''_{ss}, the relation

$$w''_{ss} = z_{ss} - \frac{z^2_{s,n+1}}{1 + z_{n+1,n+1}} \tag{10}$$

must hold. According to the theory of electric wave filters such a set of z_{ss}, $z_{s,n+1}$, $z_{n+1,n+1}$ always exists[14], since w''_{ss}, being a diagonal element of the positive real matrix, is a positive real function. However, if w''_{ss} is a constant, we must also consider an ideal transformer. Let us specially denote $z_{n+1,n+1}$ determined so as to satisfy (10) by $z^{(s)}_{n+1,n+1}$. Then the question will be raised whether we can make $z^{(s)}_{n+1,n+1}$ and $z^{(t)}_{n+1,n+1}$ $(s \neq t)$ equal.

Now method of determining z_{ss}, $z_{s,n+1}$, $z^{(s)}_{n+1,n+1}$ is as follows.[15] Let

$$w''_{ss} = -v_s/u_s, \tag{11}$$

where u_s and v_s are the polynomials in λ having no common factors; u_s is a Hurwitz polynomial (its highest coefficient being taken as 1), since w''_{ss} has no poles on the imaginary axis. Next find such a Hurwitz polynomial φ_s (with highest coefficient 1) of the lowest degree as makes f_s an even or odd function, f_s being given by

$$f_s(\lambda) f_s(-\lambda) = \tfrac{1}{2}\{u_s(\lambda)v_s(-\lambda) + u_s(-\lambda)v_s(\lambda)\}\varphi_s(\lambda)\varphi_s(-\lambda). \tag{12}$$

[12] See, for example, reference 1, p. 88.
[13] Reference 1, p. 131. Reference 5, p. 221. E. A. Guillemin, reference 9, p. 216.
[14] W. Cauer, reference 3. Reference 5, p. 389 and p. 437.
[15] Reference 14.

(Existence and uniqueness of such φ_s are established.) Then putting

$$m_s = u_s\varphi_s\psi_s^2, \quad n_s = v_s\varphi_s\psi_s^2, \tag{13}$$

where ψ_s is an arbitrary Hurwitz polynomial with highest coefficient 1, we have

$$\left.\begin{aligned} z_{ss} &= -\frac{n_s(\lambda) \pm n_s(-\lambda)}{m_s(\lambda) \mp m_s(-\lambda)}, \\ z_{s,n+1} &= \frac{2f_s(\lambda)\psi_s(\lambda)\psi_s(-\lambda)}{m_s(\lambda) \mp m_s(-\lambda)}, \\ z^{(s)}_{n+1,n+1} &= \frac{m_s(\lambda) \pm m_s(-\lambda)}{m_s(\lambda) \mp m_s(-)\lambda}, \end{aligned}\right\} \tag{14}$$

which satisfy (10). Here and in the following, the upper sign holds for $f_s(\lambda)$ even and the lower for $f_s(\lambda)$ odd. (The case when $m_s = 1$, i.e. $u_s\varphi_s = 1$ and $\psi_s = 1$, will be touched upon afterwards.) We shall prove that we can make $z^{(s)}_{n+1,n+1}$ equal to $z_{n+1,n+1}$ which is independent of s by the aid of proper ψ_s. From (11), referring to (12), we easily get

$$Re\ w''_{ss}(\lambda) = \frac{\pm f_s^2(\lambda)}{u_s(\lambda)u_s(-\lambda)\varphi_s(\lambda)\varphi_s(-\lambda)} \tag{15}$$

on the imaginary axis. And the condition of the rank, i.e.

$$Re\ w''_{ss}(j\omega)Re\ w''_{tt}(j\omega) = \{Re\ w''_{st}(j\omega)\}^2 \qquad (s \neq t) \tag{16}$$

shows that its left-hand side must be the square of an even rational function, since $Rew''_{st}(j\omega)$ on the right-hand side is the even rational function of $\lambda = j\omega$. Then, on putting (15) into the l. h. s. of (16), we know that f_s and f_t are simultaneously even or odd. For, if only one of f_s and f_t is odd, the l. h. s. of (16) can not be the square of an even function, because an odd polynomial has a zero of odd multiplicity at $\lambda = 0$. Therefore $f_1, f_2, \cdots, f_n$ must simultaneously be even or odd. Further by the same reason we know that when all square factors are removed from $u_s\varphi_s$ and $u_t\varphi_t$, the remaining factors in them must be common. Consequently, letting m be the least common multiple of $u_1\varphi_1, u_2\varphi_2, \cdots, u_n\varphi_n$, we may write

$$m = u_s\varphi_s\psi_s^2, \tag{17}$$

where ψ_s is a Hurwitz polynomial with highest coefficient 1. Returning to (13) we know that (17) gives $m_s = m$ which is independent of s. Accordingly from (14) we can determine $z^{(s)}_{n+1,n+1} = z_{n+1,n+1}$ which is independent of s, unless $m = 1$. In the case when $m = 1$, since every w''_{st} must be a constant, $W'' = \| w''_{st} \|$ is capable of representation by a network made of one ohmic resistance and ideal transformers by using the well-known method.[16] Now from (14) we have found $z_{ss}, z_{s,n+1}, z_{n+1,n+1}$ satisfying (10) for $s = 1, 2, \cdots, n$, i.e. the elements of the diagonal and the $(n+1)$th throw of Z to be obtained. But obviously $-z_{s,n+1}$ also satisfies (10), therefore the sign of $z_{s,n+1}$ remains for determination. (Note

[16] See for example, reference 5, p. 190.

that the sign of f_s is indeterminate in (12).) We must also note that we can use $m' = m\psi^2$ in place of m, where ψ is an arbitrary Hurwitz polynomial with highest coefficient 1 and corresponds to an all-pass network.

Though the sign of $z_{s,n+1}$ has not yet been determined, assuming it determined for the present, let us simply write it as $z_{s,n+1}$. Now we shall seek z_{st} $(s \neq t)$ for $s, t \neq n+1$. (The sign of $z_{s,n+1}$ being assumed to have been determined.) Let $k_{st}^{(i)}$ be the residue of z_{st} at its pole $\lambda = j\omega_i$(say). ($k_{st}^{(i)}$ may of course be zero for some values of s, t and i.) Since w_{ss}'' has no poles on the imaginary axis and z_{ss}, $z_{s,n+1}$, $z_{n+1,n+1}$ given by (14) satisfy (10), it follows from (10) that

$$k_{ss}^{(i)} - \frac{k_{s,n+1}^{(i)2}}{k_{n+1,n+1}^{(i)}} = 0 \quad \text{or} \quad k_{ss}^{(i)} = \frac{k_{s,n+1}^{(i)2}}{k_{n+1,n+1}^{(i)}} \tag{18}$$

for $k_{n+1,n+1}^{(i)} \neq 0$ and

$$k_{ss}^{(i)} \text{ and } k_{s,n+1}^{(i)} = 0$$

for $k_{n+1,n+1}^{(i)} = 0$. Similarly, since w_{st}'' $(s \neq t)$ has no poles on the imaginary axis, (9) requires that if $k_{n+1,n+1}^{(i)} \neq 0$,

$$k_{st}^{(i)} - \frac{k_{s,n+1}^{(i)} k_{t,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} = 0$$

or

$$k_{st}^{(i)} = \frac{k_{s,n+1}^{(i)} k_{t,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} \tag{19}$$

and if $k_{n+1,n+1}^{(i)} = 0$

$$k_{st}^{(i)} = 0.$$

The r. h. s. of (19) is known, therefore $k_{st}^{(i)}$ $(s \neq t,\ s, t \neq n+1)$ must be determined by (19) if $k_{n+1,n+1}^{(i)} \neq 0$. And, since $k_{n+1,n+1}^{(i)} = 0$ means that every $k_{st}^{(i)}$ is zero, we may assume $k_{n+1,n+1}^{(i)} \neq 0$. Determine every $k_{st}^{(i)}$ in this way. Then, if we write

$$K^{(i)} = \| k_{st}^{(i)} \| \qquad (s, t = 1, 2, \cdots, n+1)$$

and obtain $K^{(i)}$ relative to every pole of $z_{n+1,n+1}$, the $K^{(i)}$:s completely define

$$Z = \| z_{st} \| \qquad (s, t = 1, 2, \cdots, n).$$

Of course we are still assuming that the sign of $z_{s,n+1}$ has been determined.

Next we shall prove that Z is a positive real matrix. For this purpose let us discuss the properties of $K^{(i)}$. Comparing (18) with (19), we know that (19) holds irrespective of s and t, i.e.

$$k_{st}^{(i)} = \frac{k_{s,n+1}^{(i)} k_{t,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} \qquad (s, t = 1, 2, \cdots, n+1), \tag{19a}$$

which implies (18) for $s = t$. By squaring both sides of (19a) and using (18) we have

$$k_{st}^{(i)2} = \frac{k_{s,n+1}^{(i)2} k_{t,n+1}^{(i)2}}{k_{n+1,n+1}^{(i)2}} = k_{ss}^{(i)} k_{tt}^{(i)},$$

i.e.

$$\begin{vmatrix} k_{ss}^{(i)} & k_{st}^{(i)} \\ k_{ts}^{(i)} & k_{tt}^{(i)} \end{vmatrix} = 0 \qquad (s, t = 1, 2, \cdots, n+1). \quad (20)$$

Thus all 2-rowed principal minors of $K^{(i)}$ are zero. Next, using (18) and (19a) itself, we rewrite (19a) as

$$\begin{aligned} k_{st}^{(i)} &= \frac{k_{s,n+1}^{(i)} k_{t,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} \\ &= \frac{k_{s,n+1}^{(i)} k_{r,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} \cdot \frac{k_{n+1,n+1}^{(i)}}{k_{r,n+1}^{(i)2}} \cdot \frac{k_{t,n+1}^{(i)} k_{r,n+1}^{(i)}}{k_{n+1,n+1}^{(i)}} \qquad (21) \\ &= \frac{k_{sr}^{(i)} k_{tr}^{(i)}}{k_{rr}^{(i)}} \qquad (r, s, t = 1, 2, \cdots, n+1) \\ &\qquad\qquad \text{when } k_{rr}^{(i)} \neq 0, \end{aligned}$$

which of course implies (19a) for $r = n + 1$ and (20) for $s = t$. From (20) and (21) we easily know that

$$\begin{vmatrix} k_{rr}^{(i)} & k_{rs}^{(i)} & k_{rt}^{(i)} \\ k_{sr}^{(i)} & k_{ss}^{(i)} & k_{st}^{(i)} \\ k_{tr}^{(i)} & k_{ts}^{(i)} & k_{tt}^{(i)} \end{vmatrix} = 0 \qquad (r, s, t = 1, 2, \cdots, n+1). \quad (22)$$

(In the case when $k_{rr}^{(i)} = 0$, this is clear without (21) as $k_{rs}^{(i)} = 0$ follows from (20).) After all, from (20) and (22) we can conclude that when the elements, of $K^{(i)}$ satisfy (18) and are determined by (19), the rank of $K^{(i)}$ is 1.[17] Moreover, since z_{ss} $(s = 1, 2, \cdots, n+1)$ is a positive real function, $k_{ss}^{(i)} \geqq 0$, i.e. $K^{(i)}$ is a positive definite matrix of rank 1. Accordingly $Z = \| z_{st} \|$ defined by the $K^{(i)}$: s is a positive real matrix and can be represented by a reactive $2(n+1)$-terminal network. The above discussion also shows that a symmetrical matrix of rank 1 is uniquely determined by the elements of the diagonal and those of any one row, namely only their signs. This fact will be used afterwards.

Now suppose a reactive $2(n+1)$-terminal network whose impedance matrix is Z. Then terminate it in a unit resistance at end $(n+1)$ and find the impedance matrix with respect to end $1, 2, \cdots, n$ as in (5). Let this matrix be

$$W'''(\lambda) = \| w_{st}''' \| \qquad (s, t = 1, 2, \cdots, n).$$

Then by Theorem 1, W''' is, like W'', a positive real matrix whose real part matrix is of rank 1 on the imaginary axis, and the diagonal elements of W''' are equal to those of W'' by (10). Here we shall define the sign of $z_{s,n+1}$ reserved for determination. Since $w_{ss}''' = w_{ss}''$ $(s = 1, 2, \cdots, n)$, we have for example

$$\{Re\ w_{sn}'''\}^2 = Re\ w_{ss}'''\ Re\ w_{nn}''' = \{Re\ w_{sn}''\}^2 \qquad (23)$$

[17] Reference 11.

on the imaginary axis. (Hereafter we shall frequently omit the term "on the imaginary axis".) And since w'''_{sn} is given by

$$w'''_{sn} = z_{sn} - \frac{z_{s,n+1}\, z_{n,n+1}}{1 + z_{n+1,n+1}},$$

we have

$$Re\ w'''_{sn} = -\frac{z_{s,n+1}\, z_{n,n+1}}{1 - z^2_{n+1,n+1}} \qquad (s = 1, 2, \cdots, n).$$

From (23) and the above we know that when we choose the sign of $z_{s,n+1}$ ($s = 1, 2, \cdots, n$) properly, we may make Rew'''_{sn} equal to Rew''_{sn}, i.e. we may have the relations

$$Re\ w''_{sn} = -\frac{z_{s,n+1}\, z_{n,n+1}}{1 - z^2_{n+1,n+1}} \qquad (s = 1, 2, \cdots, n). \quad (23a)$$

For this purpose, it is sufficient that we first designate the sign of $z_{n,n+1}$ at will and thereafter determine the sign of $z_{s,n+1}$ ($s = 1, 2, \cdots, n-1$) so as to satisfy (23a). Thus we can determine Z completely. In this case, to the opposite sign of $z_{n,n+1}$ would correspond the opposite sign of $z_{s,n+1}$ ($s = 1, 2, \cdots, n-1$). That is, the elements of the $(n+1)$th row except $z_{n+1,n+1}$ can only simultaneously change their signs, the sign of $z_{n+1,n+1}$ of course being fixed. Therefore $k^{(i)}_{st}$ ($s \neq t$, $s, t \neq n+1$) is independent of the sign of $z_{s,n+1}$ as (19) clearly shows. Hence indeterminateness is only the polarity of end $(n+1)$.

Now that we have determined Z, all that remains is to prove $W''' = W''$. As above mentioned, both ReW''' and ReW'' are of rank 1 and, moreover, their elements of the diagonal and the nth row have been made equal mutually. Accordingly we must have

$$Re\ W'''(j\omega) \equiv Re\ W''(j\omega).$$

Furthermore both W''' and W'' are the positive real matrices whose elements have no poles on the imaginary axis, hence we have[18]

$$W''' \equiv W''.$$

In other words, we know that W'' has a network representation. Now connect the reactive $2n$-terminal network corresponding to W' and the reactive $2(n+1)$-terminal network corresponding to W'' in series[19] so as to make one reactive $2(n+1)$-terminal network. Then this network is the one stated in the theorem. Replacing $n+1$ by n in the theorem, we may easily presume:

THEOREM 3. A reactive $2n$-terminal network is characterized by $\frac{1}{2}n(n-1)$ functions $w_{st}(s, t = 1, 2, \cdots, n-1)$ and an arbitrary Hurwitz polynomial ψ

[18] A real function which is analytic in the right-half plane, including the boundary, is completely specified there by its real part on the boundary. See for example, reference 1, p. 145 and W. Cauer, E. N. T. 17, 17 (1940).

[19] In the network interconnection, use of ideal transformers is usually necessary.

excepting the polarity of end n, w_{st} being the element of a positive real matrix whose real part matrix is of rank 1 on the imaginary axis.

The result is not necessarily self-evident, but, since it has no direct relation to the subject of this paper, the proof is here omitted. This theorem shows a basic property of reactive networks. That is, a reactive $2n$-terminal network is determined by $\frac{1}{2}n(n-1)+1=l$ (put) or by $n-1$ less functions than the ordinary networks. As $n=1$ gives $l=1$, the above expression applies to the two-terminal networks. As to the four-terminal networks, putting $n=2$, we of course have $l=2$, which is the well-known result.[20]

We interpose here a lemma concerning the decomposition of the matrix.

LEMMA. A positive definite matrix of rank r is expressed as the sum of r positive definite matrices of rank 1.

For future reference we shall consider the lemma in some detail, though its proof is not difficult. The decomposition is not unique, but we have merely to find one method of decomposition. Write the given matrix as

$$A = \| a_{st} \| \quad (s, t = 1, 2, \cdots, n)$$

and consider the transformation of A into a diagonal matrix. As A is of rank r, at least one r-rowed principal minor is not zero. We may assume that this non-vanishing minor stands in the upper left-hand corner of A. By multiplying the first row and column by proper constants and then symmetrically adding them to the other rows and columns, we may have

$$B^{(1)} = \| b_{st}^{(1)} \|$$

where

$$b_{1t}^{(1)} = 0 \ (t \neq 1) \text{ and } b_{11}^{(1)} = a_{11} .$$

Further, transform $B^{(1)}$ with respect to its second row and column as before. Then we may have

$$B^{(2)} = \| b_{st}^{(2)} \|$$

where

$$b_{1t}^{(2)} = 0 \ (t \neq 1), \qquad b_{2t}^{(2)} = 0 \ (t \neq 2),$$
$$b_{11}^{(2)} = b_{11}^{(1)} = a_{11} , \qquad b_{22}^{(2)} = b_{22}^{(1)}.$$

Thus after the rth transformation we may have a diagonal matrix

$$B^{(r)} = \| b_{st}^{(r)} \|$$

where

$$b_{st}^{(r)} = 0 \ (s \neq t), \qquad b_{ss}^{(r)} > 0 \ (s \leqq r), \qquad b_{ss}^{(r)} = 0 \ (s > r),$$

for $B^{(r)}$ must also be a positive definite matrix of rank r. (Though the nth transformation is not necessary for $r=n$, we write the final matrix as $B^{(n)}$ for convenience.) Then decomposing $B^{(r)}$, we write

$$B^{(r)} = \sum_{p=1}^{r} C^{(p)}, \qquad C^{(p)} = \| c_{st}^{(p)} \|,$$

[20] Reference 3 and 14.

where elements of $C^{(p)}$ are all zero except $c_{pp}^{(p)} = b_{pp}^{(r)}$, $C^{(p)}$ being clearly a positive definite matrix of rank 1. The inverse of the transformation from A into $B^{(r)}$ will transform $C^{(p)}$ into $A^{(p)}$ (put). Then we have

$$A = \sum_{p=1}^{r} A^{(p)}, \tag{24}$$

where $A^{(p)}$ is a positive definite matrix of rank 1. Hence the lemma is established. Note that the elements of $A^{(p)}$ are zero except those of the p-rowed principal minor standing in the lower right-hand corner. Now let us apply the lemma to the positive real matrix.

THEOREM 4. A positive real matrix is expressed as the sum of positive real matrices whose real part matrices are of rank 1 or 0 on the imaginary axis.

As to the positive real matrix of order 2, the corresponding result was found by C. M. Gewertz.[21] Let

$$W = \| w_{st} \| \ (s, t = 1, 2, \cdots, n)$$

be the given positive real matrix. Then as in (8) we may write

$$W = W' + W''$$

where W' corresponds to the principal parts of the poles of w_{st} on the imaginary axis and both W' and W'' are the positive real matrices. Since ReW' is clearly of rank 0, we have only to study the decomposion of W''. We can apply the proof of the lemma to this case with some modification. Consider, for example, the transformation of ReW'' corresponding to that of A into $B^{(1)}$ in the lemma. In this case, the multiplication factors in the transformation will be

$$-Re\, w_{s1}''(j\omega)/Re\, w_{11}''(j\omega) \ (s = 2, 3, \cdots, n)$$

and the (s, t)-element after the transformation will be written as

$$u_{st} = Re\, w_{st}'' - \frac{Re\, w_{s1}'' Re\, w_{1t}''}{Re\, w_{11}''} \text{ (excepting } u_{11} = Re\, w_{11}'')$$

which is clearly zero for $s = 1$ or $t = 1$. Here we note that $U = \| u_{st} \|$ can be the real part matrix of a positive real matrix, which is shown as follows. In the first place, u_{st} can be the real part of a real function which is analytic in the right-half plane, including the boundary, because it satisfies the following two conditions. First, u_{st} is a function of ω^2, since Rew_{st}'' is so. Second, u_{st} has no poles for ω real. To prove this, we have only to examine the zeros of Rew_{11}'', because Rew_{st}'' $(s, t = 1, 2, \cdots, n)$ has no poles for ω real. Let O_{st} be the multiplicity of a zero of Rew_{st}''. Then from

$$Re\, w_{11}'' Re\, w_{ss}'' - \{Re\, w_{s1}''\}^2 \geqq 0,$$
$$Re\, w_{11}'' Re\, w_{tt}'' - \{Re\, w_{1t}''\}^2 \geqq 0,$$

we have

$$O_{11} + O_{ss} - 2\, O_{s1} \leqq 0,$$
$$O_{11} + O_{tt} - 2\, O_{1t} \leqq 0,$$

[21] Reference 1, p. 166.

whence

$$O_{s1} + O_{1t} - O_{11} \geqq \tfrac{1}{2}(O_{ss} + O_{tt}) \geqq 0.$$

Therefore Rew''_{s1} Rew''_{1t}/Rew''_{11} and accordingly u_{st} has no poles at zeros of Rew''_{11}. Thus u_{st} can be the real part of a real function which is analytic in the right-half plane, including the boundary, and is determined uniquely and algebraically from u_{st}.[22] Moreover, since positive definiteness is kept when we transform A into $B^{(1)}$, U determines a positive real matrix whose elements have no poles on the imaginary axis. After all, corresponding to (24) ReW'' may be written as

$$Re\, W'' = \sum_{p=1}^{r} Re\, W''^{(p)} \tag{25}$$

where r is the rank of ReW''; $ReW''^{(p)}$ is of rank 1 and can be regarded as the real part matrix of a positive real matrix $W''^{(p)}$ to be determined by $ReW''^{(p)}$. On obtaining $W''^{(p)}$ from $ReW''^{(p)}$, from (25) we must have

$$W'' = \sum_{p=1}^{r} W''^{(p)}$$

since elements of W'' and $\sum W''^{(p)}$ have no poles on the imaginary axis.[23] Consequently we have

$$W = W' + \sum_{p=1}^{r} W''^{(p)} \tag{26}$$

which proves the theorem.

In the foregoing we have made complete preparations for $2n$-terminal network synthesis. That is, from Theorem 2 and Theorem 4 we can easily deduce the following theorem:

THEOREM 5. A positive real matrix of order n is represented by a $2n$-terminal network made of at most n networks, i.e. the 4-, 6-, $\cdots$, $2(n+1)$-terminal reactive network in series, each of which is terminated in a resistance at one end.

Write the given matrix as

$$W = W' + \sum_{p=1}^{n} W''^{(p)}$$

the rank of ReW being assumed to be n; W' corresponds to a reactive $2n$-terminal network and $W''^{(p)}$ can be represented by using a reactive $2(p+1)$-terminal network, since $W''^{(p)}$ is essentially a positive real matrix of order p. Therefore, if we involve the reactive $2n$-terminal network corresponding to W' in the reactive $2(n+1)$-terminal network corresponding to $W''^{(n)}$, we shall at once have the whole network representation made of n networks in series. When the rank of ReW is $< n$, n networks are of course not necessary.

Now let us study the relation between Theorem 5 and Gewertz's theory of four-terminal network synthesis. The essential part of the latter may be said to consist in finding the network representation of such a positive real matrix of order 2 as has the property that the determinant of the real parts of its elements is equal to zero, i.e. the real part matrix is of rank 1, along the whole axis

[22] Reference 18.
[23] Reference 18.

of imaginaries.[24] That is, Gewertz reduced such a matrix to a positive real function (including a constant) by the continued-fraction type development corresponding to a reactive 6-terminal network. Then realizing the above positive real function into a two-terminal network by Brune's method,[25] he completed the whole four-terminal network. But at present we know that a positive real function can be realized by using only one resistance. Therefore Gewertz's method may be said to show that the positive real matrix of the above kind can be realized into a network containing only one resistance. Thus his network may be regarded as a particular case of the four-terminal networks given by Theorem 2. Moreover independently of his theory we can derive his network from the impedance matrix of a reactive 6-terminal network determined by Theorem 2; the actual procedure, however, is here omitted.

As we know that the impedance matrix must be a positive real matrix, Theorem 5 may also be stated as:

THEOREM 5, COROLLARY. Every $2n$-terminal network is represented by a network containing only reactive elements except n resistances.

This is an extension of the well-known fact about two-terminal networks.

We have found that every positive real matrix has a network representation, which leads to the general conclusion:

THEOREM 6. The necessary and sufficient condition to be satisfied by the impedance matrix $W(\lambda)$ of a finite passive $2n$-terminal network with lumped linear elements is: $W(\lambda)$ is a positive real matrix of order n whose elements are the rational functions of λ.

3. Other problems. So far we have concerned ourselves with networks having impedance matrices. But clearly there exist networks which can not be defined by the impedance matrices as we know in the case of four-terminal networks. Such networks, however, are the combinations of ideal transformers and networks having the impedance matrices. Therefore we need no special discussions on this point.

Next let us consider the equivalent networks resulting directly from our theory. The transformation in Theorem 4, generally speaking, corresponds to the transformation of the quadratic form into the normal form, therefore many kinds of equivalent networks may occur. As mentioned in Theorem 2, we can adopt $m\psi^2$ in place of m, which again gives equivalent networks. Furthermore even when Z has been determined, methods of synthesis of reactive networks are various. This also gives rise to equivalent networks. Changing the point of view, we can construct the network from the admittance matrix to be obtained by inverting the impedance matrix at various stages of synthesis.

Now we shall consider the minimum number of resistances necessary for $2n$-terminal network synthesis. In the case of the four-terminal network, two resistances are generally needed as shown by Theorem 1 and clearly sufficient by Theorem 5. In the case of the $2n$-terminal network, Theorem 4 shows that

[24] Reference 21.
[25] Reference 6.

r resistances are sufficient. Conversely we can also prove that r resistances are necessary.

Finally, networks obtained by Theorem 5 as well as Gewertz's four-terminal network generally contains superfluous elements; four- or more-terminal networks which contain no superfluous elements as Brune's two-terminal network are to be given by another paper[26] of the author.

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SOME NOTES ON INTEGRALS INVOLVING BESSEL FUNCTIONS

By Yudell L. Luke

In a number of applied problems, the results depend on integrals involving Bessel functions. It is the purpose of this note to develop some recursion formulae and power series expressions which can be used to compute the integrals. Finally the analysis is applied to some questions which appear to have been overlooked in the literature.

In the field of aerodynamics, Possio[1] [1] has shown that $\int_0^x e^{iu} Z_0(\lambda u)\, du$ plays a fundamental role in the calculation of flutter coefficients in subsonic flow[2]. In this problem $Z_0(x)$ represents a Bessel function of the first or second kind. Schwarz [2] made a thorough study of the above integral from the viewpoint of analysis and prepared an extensive table of these functions. v. Borbely [3] extended the theory of Possio [4] concerning supersonic flow and showed that $\int_0^x e^{iu} u^\nu J_0(\lambda u)\, du$, $\nu = 0, 1, 2, 3$, enter into formulae describing the airforces acting on a harmonically oscillating wing. In this paper, v. Borbely tabulates values of these integrals. Bennet [5] and Rice [6] have stated some results in electric circuit theory which depend on $\int_0^x e^{-u} I_0(\lambda u)\, du$ and this integral has been tabulated by the latter author. Pidduck [7] has studied the expression $\int_0^x e^{ku} u^\nu J_0(\lambda u)\, du$ which arises in the theory of currents in an aerial parallel to the earth. When $\nu = 1$, Brown [8] and Bordoni [9] have applied the integral to a problem in accoustics. Finally, King [10] employs $\int_0^x e^u K_0(u)\, du$ in a heat problem. The references given are by no means exhaustive, but serve to emphasize the importance of the integrals to be discussed here. For other types of integrals, the reader is referred to the Index of Mathematical Tables [11] and the Guide to Tables of Bessel Functions [12].

In this note, we consider the expression

$$G(\nu, n, x, \lambda) = \int_0^x e^{ku} u^\nu Z_n(\lambda u)\, du \tag{1}$$

where $Z_n(x)$ represents any of the Bessel functions of the first three kinds or the modified Bessel functions. In the analysis, the well known formulae

$$xZ_n'(x) = nZ_n(x) - axZ_{n+1}(x) \tag{2}$$

$$xZ_n'(x) = bxZ_{n-1}(x) - nZ_n(x) \tag{3}$$

are needed where the values of a and b are ± 1. Thus if $Z_n(x) = K_n(x)$, then $a = 1$ and $b = -1$.

[1] Numbers in brackets refer to bibliography at end of paper.

[2] Notation employed is not necessarily that of the authors quoted.

Integrating (1) by parts and employing (2), one obtains

$$kG(\nu, n, x, \lambda) = e^{ku}u^{\nu}Z_n(\lambda u)\Big|_0^x - (\nu + n)G(\nu - 1, n, x, \lambda) + a\lambda G(\nu, n + 1, x, \lambda) \tag{4}$$

Replacing n by $n + 1$ in (1), integrating by parts and employing (3), there results

$$kG(\nu, n + 1, x, \lambda) = e^{ku}u^{\nu}Z_{n+1}(\lambda u)\Big|_0^x - (\nu - n - 1)G(\nu - 1, n + 1, x, \lambda) - b\lambda G(\nu, n, x, \lambda) \tag{5}$$

If ν is replaced by $\nu - 1$ in (4) and these two equations are combined with (5), then

$$(k^2 + ab\lambda^2)G(\nu, n, x, \lambda) = [a\lambda e^{ku}u^{\nu}Z_{n+1}(\lambda u) + (\nu - 1 - n)e^{ku}u^{\nu-1}Z_n(\lambda u) + ke^{ku}u^{\nu}Z_n(\lambda u)]_0^x - k(2\nu - 1)G(\nu - 1, n, x, \lambda) + [n^2 - (\nu - 1)^2]G(\nu - 2, n, x, \lambda) \tag{6}$$

If, in (5), $n - 1$ is substituted for n and this equation is combined with (4), then

$$a\lambda(n - \nu)G(\nu, n + 1, x, \lambda) = -2n[e^{ku}u^{\nu}Z_n(\lambda u)]_0^x + 2nkG(\nu, n, x, \lambda) + b\lambda(n + \nu)G(\nu, n - 1, x, \lambda) \tag{7}$$

Utilising (4)–(7), it can be seen that if $G(0, 0, x, \lambda)$ is known, then $G(\nu, n, x, \lambda)$ is also known and so these equations constitute a recursive scheme for determining the latter function. We now compute $G(n, n, x, \lambda)$ in a particular case.

In a number of applications $k^2 + ab$ vanishes and this assumption is retained in the sequel. It can be shown [13] that

$$Z_n(\lambda u) = \lambda^n \sum_{m=0}^{\infty} \frac{(-1)^{\delta m}(1 - \lambda^2)^m u^m}{m!\, 2^m} Z_{m+n}(u) \tag{8}$$

In the reference cited this formula is given for Bessel functions of the first three kinds only in which case $\delta = 0$. A straight-forward reduction shows that (8) also holds for the modified Bessel functions, where for $I_n(x)$, $\delta = 1$ and for $K_n(x)$, $\delta = 0$. λ is unrestricted for Bessel Functions of the first kind while $|1 - \lambda^2| < 1$ for those of the second and third kinds.

If $\lambda = 1$ and $\nu = n + 1$, then (6) reduces to

$$G(n, n, x, 1) = \frac{1}{k(2n + 1)}[ae^{ku}u^{n+1}Z_{n+1}(u) + ke^{ku}u^{n+1}Z_n(u)]_0^x \tag{9}$$

Combining (1) and (8), it follows that

$$G(\nu, n, x, \lambda) = \lambda^n \sum_{m=0}^{\infty} \frac{(-1)^{\delta m}(1 - \lambda^2)^m}{m!\, 2^m} G(\nu + m, n + m, x, 1) \tag{10}$$

Put $\nu = n$, then (9) and (10) yield[3]

$$G(n, n, x, \lambda) = \frac{\lambda^n}{k} \sum_{m=0}^{\infty} \frac{(-1)^m (1 - \lambda^2)^m}{m!\, 2^m (2m + 1)} [ae^{ku} u^{n+m+1} Z_{n+m+1}(u) + ke^{ku} u^{n+m+1} Z_{n+m}(u)]_\epsilon^x \tag{11}$$

The lower limit of the integral (1) has purposely been taken as ϵ and not zero since the functions $Y_n(X)$ and $K_n(X)$ are singular at the origin. Letting $\epsilon \to 0$ causes no difficulty provided certain restrictions are placed on ν and n. Thus if k is imaginary, the real part of the integral converges if $\nu \geq n$ and the imaginary part of the integral converges if $\nu \geq n - 1$.

For an application of the analysis presented, consider (9). Let $k = i = \sqrt{-1}$, $Z_n(x) = Y_n(x)$. Whence $a = b = 1$. Let $\epsilon \to 0$. Then

$$\int_0^x e^{iu} u^n Y_n(u)\, du = \frac{e^{ix} x^{n+1}}{(2n + 1)} [Y_n(x) - iY_{n+1}(x)] - \frac{i}{\pi} \frac{2^{n+1} n!}{(2n + 1)} \tag{12}$$

Similarly,

$$\int_0^x e^u u^n K_n(u)\, du = \frac{e^x x^{n+1}}{(2n + 1)} [K_n(x) + K_{n+1}(x)] - \frac{2^n n!}{(2n + 1)} \tag{13}$$

and if $n = 0$,

$$\int_0^x e^u K_0(u)\, du = xe^x [K_0(x) + K_1(x)] - 1 \tag{14}$$

which is the integral due to King [10]. He constructed a table of this integral using Euler's formula and power series expansions since the value of the integral as given above was unknown. King wrote the asymptotic expansion of the integral except for a constant and concluded from some numerical computations that the constant is -1. In this regard, Bateman and Archibald [12] have theoretically confirmed the value of this constant, but they too seem to have overlooked the fact that the intergral is expressible in terms of tabulated functions.

For another application, we consider a result of Bateman and Archibald [12] that the constant term in the asymptotic expansion of $\int_0^x e^{-u} I_n(u)\, du$ is $-n$ when $n > -1$. Using the preceding analysis, we can obtain an alternative proof of this for n an integer by expressing the above integral in terms of well known functions. Let $Z_n(x) = I_n(x)$, whence $a = -1$ and $b = 1$. Also $\epsilon = 0$ and $k = -1$. If $n = 0$ in (9), then

$$\int_0^x e^{-u} I_0(u)\, du = xe^{-x} [I_0(x) + I_1(x)] \tag{15}$$

Employing the divergent series for $I_0(x)$ and $I_1(x)$ it is seen that the integral on the left has no term independent of x in its asymptotic expansion. If $n = \nu = 0$ and $\lambda = 1$ in (4), then

$$\int_0^x e^{-u} I_1(u)\, du = \int_0^x e^{-u} I_0(u)\, du + e^{-x} I_0(x) - 1 \tag{16}$$

[3] Formulae (6) and (11) are generalizations of expressions due to v. Borbely [3].

and so for $n = 1$, the theorem is true[4]. Put $\nu = 0$ and $\lambda = 1$ in (7). Then

$$\int_0^x e^{-u} I_{n+1}(u)\,du = 2e^{-x} I_n(x) + 2\int_0^x e^{-u} I_n(u)\,du - \int_0^x e^{-u} I_{n-1}(u)\,du \qquad (17)$$

and the result follows by induction. A more explicit formula which can be derived from (17) is

$$\int_0^x e^{-u} I_n(u)\,du = xe^{-x}[I_0(x) + I_1(x)] + n[e^{-x} I_0(x) - 1] + 2e^{-x}\sum_{s=1}^{n-1}(n-s) I_s(x) \qquad (18)$$

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[4] Equations (15) and (16) were first derived by Bennet [5] using relations between the hypergeometric and Bessel functions.

ON THE EXTENSION OF SOME LOMMEL INTEGRALS TO STRUVE FUNCTIONS WITH AN APPLICATION TO ACOUSTIC RADIATION*

By C. W. Horton

1. Introduction. In this paper several integrals involving Struve functions are investigated. In this connecton the integrals $\int^z z^\nu J_\nu(z)\,dz$ and $\int^z z^\nu H_\nu(z)\,dz$ occur repeatedly. For this reason the properties of these functions, tentatively designated $C_\nu(z)$ and $D_\nu(z)$, are studied. In a companion paper they are tabulated for ν equal to 0(1)4 and z equal to 0(0.1)10. It is realized that the notations $C_n(z)$ and $D_n(z)$, as well as nearly every other letter in the English alphabet, have been used by Watson in other connections.

2. A Pair of New Functions. A pair of functions $C_\nu(z)$ and $D_\nu(z)$ are introduced by the following equations

$$C_\nu(z) = \int_0^z z^\nu J_\nu(z)\,dz \qquad R(\nu) > -1/2 \quad (1)$$

$$D_\nu(z) = \int_0^z z^\nu H_\nu(z)\,dz \qquad R(\nu) > -1. \quad (2)$$

These functions can be defined more generally by the pair of infinite series

$$C_\nu(z) = \sum_{m=0}^{\infty} \frac{(-)^m 2^{1+\nu}(z/2)^{2\nu+2m+1}}{m!(2\nu+2m+1)\Gamma(\nu+m+1)} \quad (3)$$

$$D_\nu(z) = \sum_{m=0}^{\infty} \frac{(-)^m 2^{1+\nu}(z/2)^{2\nu+2m+2}}{(2\nu+2m+2)\Gamma(m+3/2)\Gamma(\nu+m+3/2)}. \quad (4)$$

These series converge for all finite values of z and therefore define functions of z which are analytic at all finite points except possibly at the origin where a pole may occur.

When the arguments of C_ν and D_ν are pure imaginary, one obtains the following formulae for the modified Bessel and Struve functions.

$$C_\nu(iz) = i^{2\nu+1}\int_0^z z^\nu I_\nu(z)\,dz \quad (5)$$

$$D_\nu(iz) = (-)^{\nu+1}\int_0^z z^\nu L_\nu(z)\,dz \quad (6)$$

The function $C_0(z)$ has been tabulated by Lowan and Abramowitz[1] to $10D$, $x = 0(.01)10$, while the function $D_0(z)$ has been tabulated (but unpublished) by

* The work described in this paper was done at the Defense Research Laboratory under the sponsorship of the Bureau of Ordnance, Navy Department, Contract NOrd-9195.

[1] Lowan, A. N., and Abramowitz, M., J. Math. Phys., **22**, 3–12, 1943.

Wrench[2] to $8D$, $z = 0(.1)10$. The function $-iC_0(iz)$ is tabulated by R. Müller[3] to $6S$, $z = 0(0.1)5.2(0.2)16.0$.

The integral for $C_\nu(z)$ is evaluated by Watson[4] in terms of Lommel functions. A more workable integration for $C_\nu(z)$ is given by McLachlan and Meyers[5]. They find

$$C_\nu(z) = 2^{\nu-1}\Gamma(1/2)\Gamma(\nu + 1/2)z\{J_\nu(z)H_{\nu-1}(z) - H_\nu(z)J_{\nu-1}(z)\}. \tag{7}$$

Despite this integral, it seems desirable to retain the notation $C_\nu(z)$ in view of the parallelism with $D_\nu(z)$ which cannot be integrated.

It is easily shown that the function $C_\nu(z)$ has the following properties.

$$C_{\nu+1}(z) = (2\nu + 1)C_\nu(z) - z^{\nu+1}J_\nu(z) \tag{8}$$

$$C''_\nu(z) = zC'_{\nu-1}(z) = z^\nu J_{\nu-1}(z) \tag{9}$$

$$2\int^z C_\nu(z)z\,dz = z^2C_\nu(z) - zC'_{\nu+1}(z) + C_{\nu+1}(z) \tag{10}$$

$$\int^z C'_\nu(z)z\,dz = C'_{\nu+1}(z) \tag{11}$$

$$\int^z C'_\nu(z)z^2\,dz = zC'_{\nu+1}(z) - C_{\nu+1}(z) \tag{12}$$

$$\int^z C''_\nu(z)z\,dz = zC'_\nu(z) - C_\nu(z) \tag{13}$$

$$C_{1/2}(z) = (2/\pi)^{1/2}(1 - \cos z). \tag{14}$$

Since $J_\nu(z) = z^{-\nu}C'_\nu(z)$, it is easy to arrive at the following third order equation for $C_\nu(z)$.

$$zC'''_\nu(z) + (1 - 2\nu)C''_\nu(z) + zC'_\nu(z) = 0 \tag{15}$$

The analogous formulae for $D_\nu(z)$ are:

$$D_{\nu+1}(z) = (2\nu + 1)D_\nu(z) - zD'_\nu(z) + z^{2\nu+2}/(\nu + 1)2^{\nu+1}\Gamma(1/2)\Gamma(\nu + 3/2) \tag{16}$$

$$D''_\nu(z) = zD'_{\nu-1}(z) = z^\nu H_{\nu-1}(z) \tag{17}$$

$$2\int^z D_\nu(z)z\,dz = z^2D_\nu(z) - zD'_{\nu+1}(z) + D_{\nu+1}(z) \tag{18}$$

$$\int^z D'_\nu(z)z\,dz = D'_{\nu+1}(z) \tag{19}$$

$$\int^z D'_\nu(z)z^2\,dz = zD'_{\nu+1}(z) - D_{\nu+1}(z) \tag{20}$$

[2] Wrench, John W., Jr., Math Tab. and Aids. Comp., III, 66, 1948.
[3] Müller, R., Zeit. F. Angew. Math. u. Mech., 19, 36–54, 1939.
[4] Watson, G. N., A Treatise on Bessel Functions, p. 350, Macmillan Co., New York, 1945.
[5] McLachlan, N. W., and Meyers, A. L., Phil. Mag. (7), 21, 437–448, 1936.

$$\int^z D''_\nu(z)z\,dz = zD'_\nu(z) - D_\nu(z) \tag{21}$$

$$D_{1/2}(z) = (2/\pi)^{1/2}(z - \sin z). \tag{22}$$

The differential equation satisfied by $D_\nu(z)$ is

$$zD'''_\nu(z) + (1 - 2\nu)D''_\nu(z) + zD'_\nu(z) = z^{2\nu}/2^{\nu-1}\Gamma(1/2)\Gamma(\nu + 1/2). \tag{23}$$

3. Integrals Involving One Function. By following a procedure used by Watson[6], it is possible to arrive at the relationship

$$\int^z [\{B(z) + (2\nu + 1)B'(z)/z + B''(z)\}H_\nu(z) + B'(z)(z/2)^\nu/\Gamma(1/2)\Gamma(\nu + 3/2)]z^{\nu+1}\,dz = z^{\nu+1}\{B'(z)H_\nu(z) + B(z)H_{\nu+1}(z)\}. \tag{24}$$

As a special case, let $B(z) = z^{\mu-\nu}$. Then one arrives at the useful reduction formula

$$\int^z z^{\mu+1}H_\nu(z)\,dz = -(\mu^2 - \nu^2)\int^z z^{\mu-1}H_\nu(z)\,dz - (\mu - \nu)z^{\mu+\nu+1}/2^\nu(\mu + \nu + 1)\Gamma(1/2)\Gamma(\nu + 3/2) + z^\mu\{(\mu - \nu)H_\nu(z) + zH_{\nu+1}(z)\}. \tag{25}$$

If $\mu = \nu$, this reduces to the well known result $\int^z z^{\nu+1}H_\nu(z)\,dz = z^{\nu+1}H_{\nu+1}(z)$. If $\mu - \nu = 2p$, an even integer, successive applications of the formula yield $\int^z z^{\nu+1}H_\nu(z)\,dz$ which can be integrated. On the other hand, if $\mu - \nu = 2p + 1$, successive applications of the formula yield $\int^z z^\nu H_\nu(z)\,dz = D_\nu(z)$.

4. Integrals Involving a Struve and a Bessel Function. Suppose that one considers the two expressions[7] $d\{z^\rho H_\mu(z)J_\nu(z)\}/dz$ and $d\{z^\mu H_{\mu+1}(z)J_{\nu+1}(z)\}/dz$. Upon performing the differentiating and then adding, one finds

$$d\{z^\rho H_\mu(z)J_\nu(z) + z^\rho H_{\mu+1}(z)J_{\nu+1}(z)\}/dz = z^{\mu+\rho}J_\nu(z)/2^\mu\Gamma(1/2)\Gamma(\mu + 3/2) + (\rho + \mu + \nu)z^{\rho-1}H_\mu(z)J_\nu(z) + (\rho - \mu - \nu - 2)z^{\rho-1}H_{\mu+1}(z)J_{\nu+1}(z). \tag{26}$$

This may be integrated to give

$$z^\rho H_\mu(z)J_\nu(z) + z^\rho H_{\mu+1}(z)J_{\nu+1}(z) = \{2^{-\mu}/\Gamma(1/2)\Gamma(\mu + 3/2)\}\int^z z^{\mu+\rho}J_\nu(z)\,dz + (\rho + \mu + \nu)\int^z z^{\rho-1}H_\mu(z)J_\nu(z)\,dz + (\rho - \mu - \nu - 2)\int^z z^{\rho-1}H_{\mu+1}(z)J_{\nu+1}(z)\,dz. \tag{27}$$

[6] Watson, G. N., *op. cit.*, p. 132.

[7] The analysis of this section and of the following one is based upon the method used by G. N. Watson, *op. cit.*, p. 136.

If $\rho + \mu + \nu = 0$, eq. (27) reduces to

$$2(\mu + \nu + 1) \int^z z^{-\mu-\nu-1} H_{\mu+1}(z) J_{\nu+1}(z)\, dz$$
$$= \{2^{-\mu}/\Gamma(1/2)\Gamma(\mu + 3/2)\} \int^z z^{-\nu} J_\nu(z)\, dz \tag{28}$$
$$- z^{-\mu-\nu}\{H_\mu(z) J_\nu(z) + H_{\mu+1}(z) J_{\nu+1}(z)\}.$$

If ν is an integer n, it is easily seen that $\int_0^z z^{-n} J_n(z)\, dz = (-)^n C_{-n}(z)$.

Alternately one may set $\rho - \mu - \nu - 2 = 0$. This gives

$$2(\mu + \nu + 1) \int^z z^{\mu+\nu+1} H_\mu(z) J_\nu(z)\, dz$$
$$= z^{\mu+\nu+2}\{H_\mu(z) J_\nu(z) + H_{\mu+1}(z) J_{\nu+1}(z)\} \tag{29}$$
$$- \{2^{-\mu}/\Gamma(1/2)\Gamma(\mu + 3/2)\} \int^z z^{2\mu+\nu+2} J_\nu(z)\, dz.$$

The last integral in eq. (29) may be evaluated formally in terms of Bessel and Lommel functions[8]. McLachlan and Meyers[9] point out that this approach is not very practical, since Lommel functions are not tabulated. They remark that if $2\mu + 2$ is "an odd positive integer or an even positive integer, the integral can be reduced to the forms $\int^z z^{\nu+1} J_\nu(z)\, dz$ or $\int^z z^\nu J_\nu(z)\, dz$ by the ordinary reduction formulae." If μ is an integer, the latter case applies and one arrives at $C_\nu(z)$.

One may set $\rho = 0$ and $\mu = \nu$ in eq. (27) to give

$$\mu \int^z H_\mu(z) J_\mu(z) z^{-1}\, dz - (\mu + 1) \int^z H_{\mu+1}(z) J_{\mu+1}(z) z^{-1}\, dz$$
$$= (\tfrac{1}{2})\{H_\mu(z) J_\mu(z) + H_{\mu+1}(z) J_{\mu+1}(z)\} \tag{30}$$
$$- C_\mu(z)/2^{\mu+1}\Gamma(\tfrac{1}{2})\Gamma(\mu + \tfrac{3}{2}).$$

When μ is increased by one and the equations added for $n - 1$ times, one has, when $\mu = 0$,

$$\int^z H_n(z) J_n(z) z^{-1}\, dz = (1/n) \sum_{p=1}^{n} C_{p-1}(z)/2^p \Gamma(\tfrac{1}{2})\Gamma(p + \tfrac{1}{2})$$
$$- (1/2n)\{H_0(z) J_0(z) + H_n(z) J_n(z)\} - (1/n) \sum_{p=1}^{n-1} H_p(z) J_p(z). \tag{31}$$

5. Integrals Involving Two Struve Functions. Methods identical in form to those of the last section yield the following formulae for integrals involving the product of two Struve functions.

[8] Watson, G. N., *op. cit.*, eq. (3), p. 350.
[9] McLachlan, N. W., and Meyers, A. L., *op. cit.*, p. 442.

$$2(\mu+\nu+1)\int^z z^{-\mu-\nu-1}H_{\mu+1}(z)H_{\nu+1}(z)\,dz$$

$$= +\{2^{-\mu}/\Gamma(\tfrac{1}{2})\Gamma(\mu+\tfrac{3}{2})\}\int^z z^{-\nu}H_\nu(z)\,dz$$

$$+\{2^{-\nu}/\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{3}{2})\}\int^z z^{-\mu}H_\mu(z)\,dz \tag{32}$$

$$-z^{-\mu-\nu}\{H_\mu(z)H_\nu(z)+H_{\mu+1}(z)H_{\nu+1}(z)\}.$$

$$2(\mu+\nu+1)\int^z z^{\mu+\nu+1}H_\mu(z)H_\nu(z)\,dz = z^{\mu+\nu+2}\{H_\mu(z)H_\nu(z)+H_{\mu+1}(z)H_{\nu+1}(z)\}$$

$$-\{2^{-\mu}/\Gamma(\tfrac{1}{2})\Gamma(\mu+\tfrac{3}{2})\}\int^z z^{2\mu+\nu+2}H_\nu(z)\,dz \tag{33}$$

$$-\{2^{-\nu}/\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{3}{2})\}\int^z z^{\mu+2\nu+2}H_\mu(z)\,dz.$$

These formulae reduce the integral of the product of two Struve functions to the integrals of the separate functions. In the case that μ and ν are integers, the two integrals on the right of eq. (33) can be reduced to $D_\nu(z)$ and $D_\mu(z)$ by repeated application of the reduction formula of eq. (25).

One also obtains

$$\int^z H_n^2(z)z^{-1}\,dz = (1/n)\sum_{p=0}^{n-1}D_p(z)/2^p\,\Gamma(\tfrac{1}{2})\Gamma(p+\tfrac{1}{2})$$

$$-(1/2n)\{H_0^2(z)+H_n^2(z)\}-(1/n)\sum_{p=1}^{n-1}H_p^2(z). \tag{34}$$

6. Further Integrals Involving Struve and Bessel Functions. By means of the common method of obtaining Lommel integrals[10] one can obtain the following relationships:

$$(k^2-l^2)\int^z J_\nu(kz)H_\nu(lz)z\,dz = z\{lJ_\nu(kz)H_{\nu-1}(lz)-kJ_{\nu-1}(kz)H_\nu(lz)\}$$

$$-4(l/2k)^{\nu+1}C_\nu(kz)/\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2}) \tag{35}$$

$$(k^2-l^2)\int^z H_\nu(kz)H_\nu(lz)z\,dz = z\{lH_\nu(kz)H_{\nu-1}(lz)-kH_{\nu-1}(kz)H_\nu(lz)\}$$

$$+\{4/\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})\}\{(k/2l)^{\nu+1}D_\nu(lz)-(l/2k)^{\nu+1}D_\nu(kz)\}. \tag{36}$$

A different, but equivalent, form of eq. (35) has been given by McLachlan and Meyers[11].

[10] For example, see p. 94 of N. W. McLachlan, Bessel Functions for Engineers, Oxford University Press, 1946.

[11] McLachlan, N. W., and Meyers, A. L., *op. cit.*, eq. (37), p. 443.

Similarly, one can obtain

$$(\mu^2 - \nu^2) \int^z H_\mu(z)H_\nu(z)z^{-1}\,dz = z\{H_\nu(z)H_{\mu-1}(z) - H_\mu(z)H_{\nu-1}(z)\}$$
$$- \{2^{1-\mu}/\Gamma(\tfrac{1}{2})\Gamma(\mu + \tfrac{1}{2})\} \int^z z^\mu H_\nu(z)\,dz \qquad (37)$$
$$+ \{2^{1-\nu}/\Gamma(\tfrac{1}{2})\Gamma(\nu + \tfrac{1}{2})\} \int^z z^\nu H_\mu(z)\,dz.$$

The corresponding integral $\int^z H_\mu(z)J_\nu(z)z^{-1}\,dz$ has been evaluated by McLachlan and Meyers[12].

Equation (37) is similar to eqs. (32) and (33) in that an integral involving the product of two Struve functions is reduced to the integrals of the separate functions. These in turn may be treated with the reduction formula of eq. (25).

7. Integrals Involving Lommel Functions. Equation (3) on p. 350 of Watson's book reads

$$\int^z z^\nu J_\nu(z)\,dz = C_\nu(z) = (2\nu - 1)zJ_\nu(z)S_{\nu-1,\nu-1}(z) - zJ_{\nu-1}(z)S_{\nu,\nu}(z). \qquad (38)$$

By using the recurrence formulae eq. (7), p. 348 and eq. (3), p. 45, one may reduce this to

$$C_\nu(z) = z\{J_\nu(z)S'_{\nu,\nu}(z) - J'_\nu(z)S_{\nu,\nu}(z)\} \qquad (39)$$

from whence one finds

$$S_{\nu,\nu}(z) = J_\nu(z) \int^z \{C_\nu(z)/zJ^2_\nu(z)\}\,dz. \qquad (40)$$

By expanding the function

$$\frac{d}{dz}\{z^{1-\nu}H_{\nu-1}(z) \cdot z^\nu S_{\mu,\nu}(z)\} - (\mu + \nu - 1)\frac{d}{dz}\{z^\nu H_\nu(z) \cdot z^{1-\nu}S_{\mu-1,\nu-1}(z)\}$$

and integrating, one arrives at the relationship

$$\int^z z^\nu S_{\mu,\nu}(z)\,dz = 2^{\nu-1}\Gamma(\tfrac{1}{2})\Gamma(\nu + \tfrac{1}{2})\{zH_{\nu-1}(z)S_{\mu,\nu}(z)$$
$$- (\mu + \nu - 1)zH(z)S_{\mu-1,\nu-1}(z) + \int^z z^\mu H_\nu(z)\,dz\}. \qquad (41)$$

8. $C_3(z)$ and $D_3(z)$ expressed as definite integrals. By using Sonine's first finite integral and a variation of it given by Watson[13] one can deduce

$$C_\mu(z) = (2/\pi)^{\frac{1}{2}} z^{\mu+\frac{1}{2}} \int_0^{\pi/2} J_{\mu+\frac{1}{2}}(z \sin\theta) \sin^{\mu-\frac{1}{2}}\theta\,d\theta \qquad (42)$$

[12] McLachlan, N. W., and Meyers, A. L., *op. cit.*, eq. (38), p. 443.
[13] *op. cit.*, pp. 373 and 374, eqs. (1) and (3).

and

$$D_\mu(z) = (2/\pi)^{\frac{1}{2}} \int_0^{\pi/2} C_{\mu+\frac{1}{2}}(z \sin\theta) \sin^{-2\mu-1}\theta \, d\theta. \tag{43}$$

In eq. (42) $R^\mu > -\frac{1}{2}$ while in eq. (43) μ is unrestricted.

Meijer[14] has shown that Sonine's first finite integral is satisfied when the Bessel function is replaced by a Struve function. This result may be used to obtain

$$D_\mu(z) = (2/\pi)^{\frac{1}{2}} z^{\mu+\frac{1}{2}} \int_0^{\pi/2} H_{\mu+\frac{1}{2}}(z \sin\theta) \sin^{\mu-\frac{1}{2}}\theta \, d\theta \tag{44}$$

provided $\mu > -1$.

9. Application to a Problem in Acoustic Radiation. The functions $C_\nu(z)$ and $D_\nu(z)$ occur naturally in the following problem of acoustic radiation. Consider a semicircular disk of radius a that is mounted in an infinite stiff baffle. Suppose that each part of the plate vibrates sinusoidally with an amplitude distribution given by

$$f(r, \zeta) = C_0 r^{2\nu-1} \sin^{2\nu}\zeta \tag{45}$$

where (r, ζ) is a polar coordinate system on the disk. The radii $\zeta = \pm\pi/2$ form the bounding diameter.

Consider the plane that passes through the center of the semi-circular disk and which is perpendicular to the bounding diameter. The pressure distribution, $p(\theta)$, at points in this plane which are far from the disk may be found from[15]

$$p(\theta) \alpha \int_0^a \int_{-\pi/2}^{\pi/2} f(r, \zeta) \exp(ikr \cos\zeta \sin\theta - i\omega t) \, r dr \, d\zeta. \tag{46}$$

The constant of proportionality is chosen so that $p(0) = 1$. In this equation θ is the angle between the normal to the disk and the direction to the point of observation. The symbol k denotes $2\pi/\lambda$.

When the constant of proportionality is taken as $4\Gamma(\nu+1)/\Gamma(\frac{1}{2})\Gamma(\nu-\frac{1}{2})a^{2\nu+1}$, one may integrate eq. (43) to give

$$p(\theta) = [2^\nu(2\nu+1)\Gamma(\nu+1)\{C_\nu(ka\sin\theta) + iD_\nu(ka\sin\theta)\}/(ka\sin\theta)^{2\nu+1}]e^{-i\omega t}. \tag{47}$$

The acoustic pattern for the full circular disk may be obtained from eq. (47) by adding to itself the same expression with a minus sign attached to the $D_\nu(ka \sin\theta)$.

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[14] Meijer, C. S., Konin. Nederl. Akad. von Weten, **43**, 366–78, 1940.
[15] Jones, R. Clarke, Jour. Acoust. Soc. Amer., **16**, 147–171, 1945.

YIELD CONDITIONS IN PLANE PLASTIC STRESS*

By P. G. Hodge, Jr.**

1. Introduction. In the study of perfect plasticity, problems in plane strain have received considerable attention. The subject of plane stress, however, is intrinsically more difficult. In plane strain, the two generally accepted yield conditions coincide, while in plane stress they do not, thus leading to more duplication of effort. In plane strain, the characteristics of the stress equations exist at each point; they form an orthogonal net and are identical with the lines of maximum shearing stress and maximum shear strain. In the case of plane stress, few of these simplifications subsist.

In the present paper an attempt will be made to clarify these problems. The next two sections are primarily a review of the work done by Sokolovsky [1, 2, 3][1] and v. Mises [4]. In addition, a family of new yield conditions is proposed to facilitate the comparison of various yield conditions. In Sec. 4, the problem of a flat plate with a circular hole, subject to internal pressure on the hole and hydrostatic tension at infinity, is solved according to these conditions. A slight change in yield condition is seen to produce only a slight change in the stress distribution. In Sec. 5, the solution for a notched bar in tension is obtained. Here it is found that the precise form of the yield condition has a profound effect on the solution. A tentative explanation of this difference in behavior is advanced. Finally, in Sec. 6, the same problem is solved according to the Mises yield condition.

2. Stress equations. For a material in a state of plane stress, referred to the x, y plane, the only non-vanishing stress components are σ_x, σ_y, and τ_{xy}. For a perfectly plastic material, these must satisfy two equilibrium equations,

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0, \qquad \partial\sigma_y/\partial y + \partial\tau_{xy}/\partial x = 0 \tag{2.1}$$

and a yield condition. Thus, given suitable boundary conditions, the stress problem is statically determinate. The three stress components may be replaced by three new variables, defined by

$$\left.\begin{aligned} \sigma_x &= 2k(\omega + \chi \sin 2\theta), \qquad \sigma_y = 2k(\omega - \chi \sin 2\theta) \\ \tau_{xy} &= -2k\chi \cos 2\theta. \end{aligned}\right\} \tag{2.2}$$

Here k is the yield stress in pure shear and θ is the angle between the first shear direction[2] and the negative y axis. If σ_1 and σ_2 are the maximum and minimum tensile stresses (in the x, y plane), then

$$\omega = (\sigma_x + \sigma_y)/4k = (\sigma_1 + \sigma_2)/4k, \qquad \chi = (\sigma_1 - \sigma_2)/4k \geqq 0.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The first shear direction at a point P is defined as the direction obtained by a 45° counter-clockwise rotation of the line element in the x, y plane at P across which the tensile stress is a maximum.

The most general isotropic yield condition can be represented by

$$\chi = f(\omega). \tag{2.3}$$

The substitution of Eqs. 2.2 and 2.3 into 2.1 leads to

$$\left.\begin{aligned}(1 + f' \sin 2\theta)(\partial\omega/\partial x) - f' \cos 2\theta(\partial\omega/\partial y) \\ + 2f[\cos 2\theta(\partial\theta/\partial x) + \sin 2\theta(\partial\theta/\partial y)] = 0, \\ - f' \cos 2\theta(\partial\omega/\partial x) + (1 - f' \sin 2\theta)(\partial\omega/\partial y) \\ + 2f[\sin 2\theta(\partial\theta/\partial x) - \cos 2\theta(\partial\theta/\partial y)] = 0.\end{aligned}\right\} \tag{2.4}$$

The characteristics of Eqs. 2.4 may be defined as those curves across which ω and θ are continuous but may have discontinuous derivatives. Let

$$x = x(s), \qquad y = y(s), \qquad \omega = \omega(s), \qquad \theta = \theta(s)$$

along a curve Γ. Then along Γ

$$\frac{\partial\omega}{\partial x}\frac{dx}{ds} + \frac{\partial\omega}{\partial y}\frac{dy}{ds} = \frac{d\omega}{ds}, \qquad \frac{\partial\theta}{\partial x}\frac{dx}{ds} + \frac{\partial\theta}{\partial y}\frac{dy}{ds} = \frac{d\omega}{d\theta}.$$

The derivatives of ω and θ can exhibit a discontinuity across Γ only if the 5×4 matrix

$$M = \begin{vmatrix} 1 + f' \sin 2\theta & - f' \cos 2\theta & 2f \cos 2\theta & 2f \sin 2\theta & 0 \\ -f' \cos 2\theta & 1 - f' \sin 2\theta & 2f \sin 2\theta & - 2f \cos 2\theta & 0 \\ dx & dy & 0 & 0 & d\omega \\ 0 & 0 & dx & dy & d\theta \end{vmatrix}$$

is of rank less than 4. The determinant of the first four columns can vanish only along the curves

$$\frac{dy}{dx} = - \frac{\cos 2\theta \pm \sqrt{1 - f'^2}}{f' + \sin 2\theta}. \tag{2.5}$$

Here the upper and lower signs are said to refer to the first and second families of characteristics respectively.

A pair of first order equations is said to be hyperbolic, parabolic, or elliptic according to whether their characteristics are real and distinct, real and coincident, or imaginary, respectively; in the present case, according to whether $|f'|$ is less than, equal to, or greater than 1, respectively. For elliptic equations, little is to be gained by discussing the characteristics further. For hyperbolic or parabolic equations, it is necessary to set the four other 4×4 determinants which are obtained by omitting one column of M, equal to zero. This leads to

$$d\theta = \pm d\omega\sqrt{1 - f'^2}/2f \tag{2.6}$$

along the first and second characteristics respectively. Let

$$\left.\begin{aligned}\psi(\omega) &= \tfrac{1}{2} \sin^{-1} f'(\omega) \text{ where } -\pi/4 < \psi < \pi/4, \\ \Omega(\omega) &= \int (\sqrt{1 - f'^2}/2f)\, d\omega, \qquad \xi = \Omega + \theta, \qquad \eta = \Omega - \theta.\end{aligned}\right\} \tag{2.7}$$

Then the first and second characteristics are given by

$$dy/dx = -\cot(\theta + \psi), \qquad dy/dx = \tan(\theta - \psi), \tag{2.8}$$

respectively, while Eqs. 2.6 show that that η and ξ are constant along a first and second characteristic, respectively. Finally, the equilibrium equations become[3]

$$\left.\begin{aligned} \cos(\theta - \psi)(\partial\xi/\partial x) + \sin(\theta - \psi)(\partial\xi/\partial y) &= 0, \\ \sin(\theta + \psi)(\partial\eta/\partial x) - \cos(\theta + \psi)(\partial\eta/\partial y) &= 0. \end{aligned}\right\} \tag{2.9}$$

These equations may be solved numerically by standard methods.

If the stress equations are parabolic, then the yield condition must be

$$f(\omega) = c \pm \omega, \tag{2.10}$$

where c is a constant. The single family of characteristics are

$$\frac{dy}{dx} = \frac{-\cos 2\theta}{\pm 1 + \sin 2\theta} = \tan(\theta \mp \pi/4). \tag{2.11}$$

where the sign must conform to the one used in Eq. 2.10. In view of Eq. 2.6, θ is constant along each characteristic, so that Eq. 2.11 can be integrated:

$$y = x \tan(\theta \mp \pi/4) + H_1(\theta). \tag{2.12}$$

The substitution of Eq. 2.12 into either of Eqs. 2.4 leads finally to an expression for ω:

$$\omega = \frac{H_2(\theta)}{x \sec^2(\theta \mp \pi/4) + H_1'(\theta)} \mp c. \tag{2.13}$$

Thus the solution of a boundary value problem is reduced to the determination of H_1 and H_2.

3. Yield conditions. The most generally used yield conditions are due to Tresca [5] and Mises [6]. In terms of the present variables, Tresca's condition is

$$\chi = \tfrac{1}{2} \text{ for } |\omega| < \tfrac{1}{2}, \qquad \chi = 1 - |\omega| \text{ for } |\omega| > \tfrac{1}{2}, \tag{3.1}$$

and Mises' is

$$\chi = \sqrt{(1 - \omega^2)/3}. \tag{3.2}$$

If f' is computed for either of these conditions, it is seen that the stress equations are hyperbolic for certain combinations of stresses, and are otherwise parabolic or elliptic. Since it is generally simpler to apply numerical methods to hyperbolic equations, Mises [4] recently proposed a "parabolic" yield condition (so named from its shape in a σ_1, σ_2 plane, Fig. 1) of the form

$$\chi = (1 + \sqrt{2})/4 - (\sqrt{2} - 1)\omega^2. \tag{3.3}$$

[3] Equations 2.8 were first obtained by Sokolovsky for a particular yield condition [1].

For this law the stress equations are always hyperbolic. To investigate the effect on the solution of changing the type of yield condition, the family of "generalized Tresca" yield conditions is convenient:

$$\chi = \tfrac{1}{2} \text{ for } |\omega| < \tfrac{1}{2}, \qquad \chi = (1+\lambda)/2 - \lambda|\omega| \text{ for } |\omega| > \tfrac{1}{2}, \tag{3.4}$$

where λ is a positive constant. The type of the stress equations now depends upon whether λ is greater or less than one. Equations 3.4 reduce to the Tresca condition for $\lambda = 1$, and to the yield condition in plane strain for $\lambda = 0$.

The various yield conditions are indicated graphically in Fig. 1. It is seen that they all agree rather closely with each other. In Table 1, a summary of their salient properties is presented.

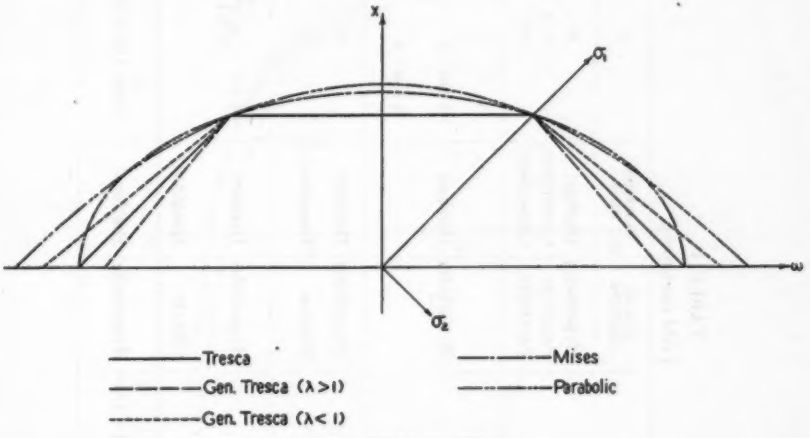

FIG. 1. Yield conditions

4. Circular hole under internal pressure. Consider a circular hole of unit radius in an infinite plate. A uniform pressure $2pk$ is applied to the contour of the hole, and the plate is subjected to a hydrostatic tension, sufficient to cause plasticity throughout, at infinity. The problem is conveniently expressed in terms of polar coordinates r and φ, the stress components being σ_r, σ_φ, and $\tau_{r\varphi}$. Due to symmetry, $\tau_{r\varphi} = 0$ and $\theta = \varphi - \pi/4$. The normal stresses are functions of r only, and are given by

$$\sigma_r = 2k[\omega - f(\omega)], \qquad \sigma_\varphi = 2k[\omega + f(\omega)]. \tag{4.1}$$

They must satisfy the equilibrium equation

$$(d\sigma_r/dr) + (\sigma_r - \sigma_\varphi)/r = [d(\omega - f)/dr] - 2f/r = 0. \tag{4.2}$$

The inverse solution of this equation may be expressed as

$$\log r = \int_{\omega_0}^{\omega} \{[1 - f'(\omega)]/2f(\omega)\}\, d\omega, \tag{4.3}$$

TABLE I

Yield conditions

YIELD CONDITION	RANGE OF ω	$\chi = f(\omega)$	TYPE OF EQUATIONS	CHARACTERISTICS	ψ	Ω
Tresca	$-\frac{1}{2} < \omega < \frac{1}{2}$ $\frac{1}{2} < \omega < 1$ $-1 < \omega < -\frac{1}{2}$	$\frac{1}{2}$ $1 - \omega$ $1 + \omega$	Hyperbolic Parabolic Parabolic	Distinct Coincident Coincident	0 $-\pi/4$ $\pi/4$	ω 0 0
Generalized Tresca $\lambda < 1$	$-\frac{1}{2} < \omega < \frac{1}{2}$ $\frac{1}{2} < \omega < \frac{1+\lambda}{2\lambda}$ $-\frac{1+\lambda}{2\lambda} < \omega < -\frac{1}{2}$	$\frac{1}{2}$ $\frac{1+\lambda}{2} - \lambda\omega$ $\frac{1+\lambda}{2} + \lambda\omega$	Hyperbolic	Distinct	0 $-\frac{1}{2}\sin^{-1}\lambda$ $\frac{1}{2}\sin^{-1}\lambda$	ω $-\frac{\sqrt{1-\lambda^2}}{2\lambda}\log\left(\frac{1+\lambda}{2\lambda} - \omega\right)$ $\frac{\sqrt{1-\lambda^2}}{2\lambda}\log\left(\frac{1+\lambda}{2\lambda} + \omega\right)$
Generalized Tresca $\lambda > 1$	$\lvert\omega\rvert < \frac{1}{2}$ $\frac{1}{2} < \lvert\omega\rvert < \frac{1+\lambda}{2\lambda}$	$\frac{1}{2}$ $\frac{1+\lambda}{2} - \lambda\lvert\omega\rvert$	Hyperbolic Elliptic	Distinct Imaginary	0 —	ω —
Mises	$\lvert\omega\rvert < \sqrt{3}/2$ $\sqrt{3}/2 < \lvert\omega\rvert < 1$	$\sqrt{\frac{1-\omega^2}{3}}$	Hyperbolic Elliptic	Distinct Imaginary	$-\frac{1}{2}\sin^{-1}\frac{\omega}{\sqrt{3(1-\omega^2)}}$ —	$\frac{1}{2}\int\frac{\sqrt{3-4\omega^2}}{1-\omega^2}\,d\omega$ —
Parabolic	$\lvert\omega\rvert < \sqrt{\frac{1+\sqrt{2}}{4(\sqrt{2}-1)\omega^2}}$	$\frac{1+\sqrt{2}}{4} - (\sqrt{2}-1)\omega^2$	Hyperbolic	Distinct	$-\frac{1}{2}\sin^{-1}[2(\sqrt{2}-1)\omega]$	$2\int\frac{\sqrt{1-4(3-2\sqrt{2})\omega^2}}{1+\sqrt{2}-4(1-\sqrt{2})\omega^2}\,d\omega$

where ω_0 is determined from the boundary condition

$$\sigma_r(1) = 2k[\omega_0 - f(\omega_0)] = -2pk.$$

Since σ_r and σ_φ are of opposite signs on the hole, the generalized Tresca yield condition in some neighborhood of the hole is $f = \frac{1}{2}$. Hence

$$\omega = \tfrac{1}{2} - p + \log r.$$

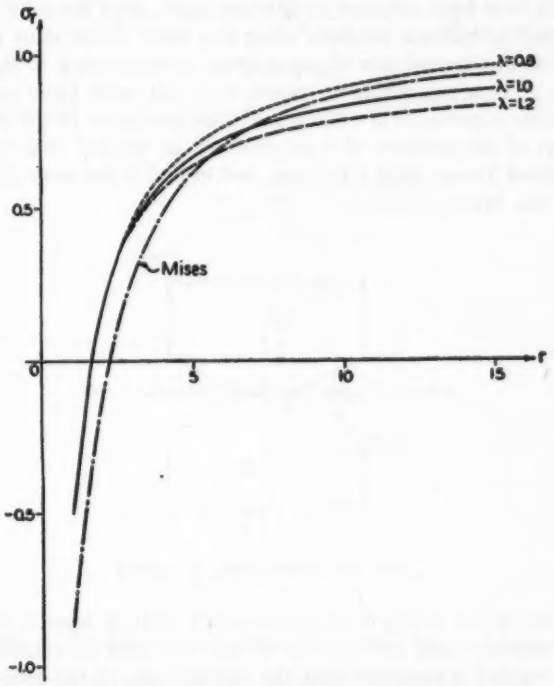

FIG. 2. Radial stresses in a plate with a circular hole, $p = 1/2$.

This solution is valid for $r < e^p$, at which point $\omega = \frac{1}{2}$. For $r > e^p$, the yield condition is $f = (1 + \lambda)/2 - \lambda\omega$, hence

$$\omega = (1/2\lambda)[1 + \lambda - (e^p/r)^{2\lambda/(1+\lambda)}].$$

The stress components are then found from Eqs. (4.1):

$$\left.\begin{aligned} &\sigma_r = 2k(-p + \log r), \qquad \sigma_\varphi = 2k(1 - p + \log r) \quad \text{for } r < e^p, \\ &\sigma_r = k\,\frac{1+\lambda}{\lambda}\,[1 - (e^p/r)^{2\lambda/(1+\lambda)}], \\ &\sigma_\varphi = k\,\frac{1+\lambda}{\lambda}\left[1 + \frac{1-\lambda}{1+\lambda}\,(e^p/r)^{2\lambda/(1+\lambda)}\right] \qquad \text{for } r > e^p. \end{aligned}\right\} \tag{4.3}$$

This solution, considered as a function of λ is regular in the neighborhood of $\lambda = 1$. Therefore, in this example, a small change in the yield condition causes only a small change in the stress solution. In Fig. 2, the values of σ_r are plotted as a function of r for $\lambda = 0.8$, 1.0, and 1.2. Also shown is the solution according to the Mises criterion [1].

5. Notched strip in tension. The solutions to many of the problems in plane strain which have been obtained to date are based upon the solution for a blunt wedge subject to uniform pressure along one face[4]. Since plane strain may be considered as the special case of plane stress corresponding to the yield condition $f(\omega) = \frac{1}{2}$, it is reasonable to expect that this same basic problem will be of considerable importance in other plane stress problems. In the present section, the solution of the problem of a notched bar in tension[5] will be obtained for the generalized Tresca yield condition, and in Sec. 6 the same problem will be solved for the Mises condition.

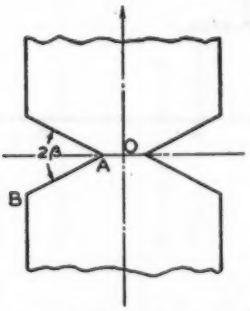

FIG. 3. Notched strip in tension

Let the bar shown in Fig. 3 be pulled with a uniform tension so that $\sigma_s = 0$. Due to symmetry, only one quarter of the bar need be considered (Fig. 4). It may be verified *a posteriore* that the correct form of the generalized Tresca yield condition is

$$f(\omega) = \omega \sin 2\psi + (1 - \sin 2\psi)/2, \tag{5.1}$$

where $\sin 2\psi = -\lambda$ is a constant. Since there are no tractions along AB, and since the stress distribution must be symmetric across $A0$, the boundary conditions are

$$\text{on } AB\text{: } \omega_1 = \tfrac{1}{2}, \qquad \theta_2 = \frac{\pi}{4} + \beta; \qquad \text{on } A0\text{: } \theta_1 = \frac{3\pi}{4}. \tag{5.2}$$

[4] The solution of this problem may be found in [7]. It is the basis of solutions to problems of drawing [8] and extrusion [9], in the indentation of a plastic mass by a rigid wedge [10], [11], and in tension of a notched bar [12].

[5] It seems more valuable to consider a tensile problem, since under compression a system may buckle rather than flow plastically. However, the tensile solution may be formally adapted to compression by obvious modifications.

In analogy with the plane strain solution, the solution is assumed to consist of three regions, as indicated in Fig. 4. Introducing polar coordinates r, φ, where r is the distance from A and φ is the angle with the negative y-axis, the solution is finally found in the form

$$\left.\begin{aligned}
\omega_1 &= [\tfrac{1}{2}\sin 2\psi]\cdot[-1+\sin 2\psi+e^{2(\pi/2-\beta)\tan 2\psi}],\\
\omega_2 &= [\tfrac{1}{2}\sin 2\psi]\cdot[-1+\sin 2\psi+e^{2(\pi/4+\varphi+\psi-\beta)\tan 2\psi}],\\
\omega_3 &= \tfrac{1}{2}, \qquad \theta_1 = 3\pi/4, \qquad \theta_2 = \pi/2+\varphi+\psi, \qquad \theta_3 = \pi/4+\beta,\\
\varphi_{12} &= \pi/4-\psi, \qquad \varphi_{23} = \beta-(\pi/4+\psi),\\
p &= [\tfrac{1}{2}\sin 2\psi]\cdot[-1+\sin 2\psi+(1+\sin 2\psi)e^{2(\pi/2-\beta)\tan 2\psi}].
\end{aligned}\right\} \quad (5.3)$$

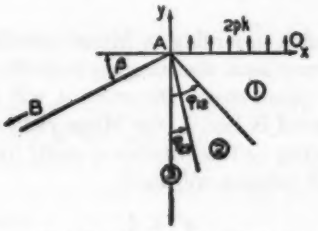

FIG. 4. Details of notched strip in tension

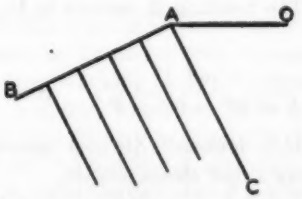

FIG. 5. Parabolic characteristics for notched bar in tension

Here φ_{12} and φ_{23} are the rays separating the regions, and p gives the tension necessary to obtain plastic flow.

Tresca's yield condition is obtained by setting $\psi = -\pi/4$ in Eq. 5.1. However, the stress distribution obtained setting $\psi = -\pi/4$ in Eqs. 5.3 does not satisfy the equilibrium equations, and hence can not be the solution. In fact, it will now be shown that according to Tresca, a continuous solution does not even exist.

The boundary condition $\sigma_n = \tau = 0$ along AB, determines a field of constant stress bounded by the characteristic AC, perpendicular to AB (Fig. 5). Thus in region $0AC$, the boundary conditions

$$\theta(x, 0) = 3\pi/4, \qquad \theta(x, -x\tan\beta) = \pi/4+\beta, \qquad \omega(x, -x\tan\beta) = \tfrac{1}{2}$$

must be satisfied. The first two boundary conditions can be satisfied only if $H_1(\theta) = 0$ in Eq. 2.12, hence Eq. 2.13 becomes

$$\omega = (1/x)H_2(\theta)\cos^2(\theta+\pi/4)+1.$$

For no value of H_2 can the third boundary condition be satisfied.

As yet it has not been possible to obtain the solution for the elliptic case ($\lambda > 1$). Neither has it been possible to prove whether or not such a solution can exist, although it can be shown that there cannot be a solution independent of r. However, even if the elliptic solution could be found, it could by no means be considered as an approximation to the hyperbolic solution (or vice versa), since, as $\lambda \to 1$ from the left, the latter solution does not approach a finite limit. Therefore, in the present example, it would not be permissible to approximate one type of yield condition by another. Lacking further examples, it seems reasonable to conclude that the good agreement in the radially symmetric case is fortuitous. Possibly it is due to the fact that one integration ($\tau_{r\varphi} = 0$) can be performed independently of the yield condition, thus reducing the problem to a single first order equation.

6. Notched bar in tension according to Mises criterion. The negative result of the preceding section may seem discouraging from the viewpoint of obtaining solutions to problems in plane stress. However, it will now be shown that for the basic solution considered in Sec. 5, the Mises yield condition is hyperbolic, so that the solution according to this criterion is easily found.

If it is assumed that the solution will show

$$\omega^2 < \tfrac{3}{4} \tag{6.1}$$

throughout the plastic domain, then the characteristics defined by Eqs. 2.8 will be real and distinct. The function Ω, defined in Eq. 2.7, may be expressed parametrically by

$$\Omega = -\tfrac{1}{2}\tan^{-1}\frac{16t^3}{1+9t^2-9t^4-t^6}, \qquad \omega = \frac{\sqrt{3}}{2}\frac{1-t^2}{1+t^2}. \tag{6.2}$$

All first characteristics which intersect $A0$ also intersect AB (Fig. 4). Since $\eta = \Omega - \theta$ is constant along a first characteristic,

$$\Omega_1 = \theta_1 + \Omega_2 - \theta_2 \simeq 1.2318 - \beta \tag{6.3}$$

along $A0$, and hence in region 1. Further, it follows from the geometry of Fig. 4, that in region 2

$$\theta_2 = \pi/2 + \varphi + \psi, \qquad \Omega_2 = \theta_2 + \Omega_3 - \theta_3 = \frac{\pi}{2} + \varphi + \psi_2 + \Omega_3 - \theta_3, \tag{6.4}$$

where φ is the polar angle from the negative y axis. The last Eq. 6.4 may be solved numerically for ω_2 to complete the solution of the problem.

In view of the boundary conditions (5.2), the inequality 6.1 is satisfied in region 3. Further, it will be satisfied in region 1 (and hence throughout the plastic domain) unless $\Omega_1 = 0$. Thus, for the particular case

$$\beta \simeq 1.2318 \text{ radians} \simeq 70.6°, \tag{6.5}$$

the solution will be parabolic in region 1. Therefore, it is only necessary to verify that a hyperbolic solution exists for values of β slightly smaller and slightly larger than 70.6°.

Since for β satisfying 6.5, $t_1 = 0$, this is conveniently done by setting $t_1 = \epsilon$ (where ϵ may be either positive or negative) and expressing the solution as a power series in ϵ. To within fourth order terms, it is found that

$$\left.\begin{aligned}
\theta_1 &= 2.36, & \theta_3 &= 2.02 + 8\epsilon^3,\\
\omega_1 &= \tfrac{1}{2}\sqrt{3}(1 - 2\epsilon^2 + 2\epsilon^4), & \omega_3 &= 0.50,\\
\Omega_1 &= -8\epsilon^3, & \Omega_3 &= -0.34,\\
\psi_1 &= (\pi/4) + 2\,|\,\epsilon\,| - 32\,|\,\epsilon^3\,|/3, & \psi_3 &= -0.17,\\
\chi_1 &= 3(\tfrac{1}{6} + \epsilon^2 - \tfrac{5}{3}\epsilon^4), & \chi_3 &= 0.50,\\
\beta &= 1.23 + 8\epsilon^3,\\
\varphi_{12} &= (\pi/2) - 2\,|\,\epsilon\,| + 32\,|\,\epsilon^3\,|/3,\\
\varphi_{23} &= 0.62 + 8\epsilon^3, \qquad p = 1.15(1 - \epsilon^4),
\end{aligned}\right\} \tag{6.6}$$

where the decimal numbers represent approximate values of fractional and irrational terms appearing in the calculations.

Equations 6.6 will represent a solution to the problem provided that the regions defined by φ_{12} and φ_{23} actually lie within the bar (Fig. 4). This will be true provided the inequalities

$$-\beta < \varphi_{23} < \varphi_{12} < \pi/2$$

are valid. For all ϵ different from zero these inequalities do, in fact, hold. For $\epsilon = 0$, $\varphi_{12} = \pi/2$, so that in this case region 1 disappears. However, since this occurs only for one particular value of β, the solution represented by regions 2 and 3 will satisfy the boundary conditions and will be hyperbolic throughout. Since Eq. 6.5 represents the only critical value of β it may be concluded that the hyperbolic solution is valid for notches of any angle.

Also of interest in Eqs. 6.6 is the fact that p is a relative maximum for $\epsilon = 0$. Thus the force required to cause flow in a notched bar is greater for a notch semi-angle of about 70.6° than for neighboring notch angles. This prediction could presumably be tested by experiment.

The value of the pressure may be compared with that predicted by the generalized Tresca yield condition. If

$$\lambda = -\sin 2\psi = 0.578,$$

the values of ψ in regions 1 and 3 will be the same as for the Mises yield condition. The substitution of this value of ψ, together with $\beta = 1.23$ into the last Eq. 5.3 leads to

$$p = 1.14,$$

in close agreement with the last Eq. 6.4.

7. Conclusions. The stress equations for a perfectly plastic material, loaded so as to be in a state of plane stress, have been obtained for an arbitrary yield

condition. For reasonable yield conditions it is found that these equations are hyperbolic in part of the range of admissible stresses, and may be hyperbolic, parabolic, or elliptic in the remainder, according to the derivative of the yield condition. Since a slight change in the yield condition may change the type of the equations, the question arises as to whether this slight change will produce only slight changes in the stress distribution. In a radially symmetric problem, this is found to be the case, but for a notched bar it does not appear to be. Therefore, in general it would seem that any such approximation must be justified by special considerations before it is advanced as a solution.

Further, it is found that for the important case of the notched bar in tension, the solution according to the Mises criterion is hyperbolic, so that, in this case at least, there is no necessity for introducing other yield conditions.

The conclusions reached here are also of interest in the axially symmetric stress problem. Ever since Hencky's paper in 1923 [13], attempts have been made to approximate the yield condition by a simple expression which would allow problems to be solved by the method of characteristics. In a recent paper [14], Symonds has shown that the axially symmetric stress equations are, in fact, elliptic throughout the plastic domain. Therefore, in consideration of the second example of this paper, it appears unlikely that a good approximation could be obtained by hyperbolic equations.

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TABLES OF INTEGRALS OF STRUVE FUNCTIONS

By Milton Abramowitz

Struve's functions $H_\nu(x)$ are of importance in the theory of diffraction and have been tabulated extensively for $\nu = 0, 1$ [1]. The integrals of these functions arose in a paper by Levine and Schwinger [2] in the process of evaluating certain definite integrals.

The series expansion of these functions is [3]

$$H_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{\nu+2m+1}}{\Gamma(m+3/2)\Gamma(m+\nu+3/2)}.$$

The modified Struve function, $L_\nu(x)$, which has the same relation to $H_\nu(x)$ as $I_\nu(x)$ has to $J_\nu(x)$, is defined by a series similar to the one given above except that all the terms are positive. The present article contains the integrals of the functions $H_\nu(x)$ and $L_\nu(x)$ for $\nu = 0, 1$ in the region $x = 0$ to $x = 10$ at intervals of .1. If we use the notation $\bar{y} = \int_0^x y(t)\,dt$ we get

$$\bar{H}_\nu(x) = \sum_{m=0}^{\infty} \frac{2(-1)^m (x/2)^{\nu+2m+2}}{(\nu+2m+2)\Gamma(m+3/2)\Gamma(m+\nu+3/2)}.$$

This expression was used to spot check the integrated values at several points in the range $x = 0$ to $x = 10$. The calculations were performed by numerical integration of the functions $H_\nu(x)$ using the values in Watson at intervals of .1. The functions $L_\nu(x)$ were obtained by numerical integration, using the results (to eight or more significant figures) previously obtained by the Computation Laboratory.

The corresponding spot checks were made for these entries using the series expansion for the integral of the modified functions. In general, interpolation with five-point Lagrangian interpolation coefficients will yield the full accuracy of the tables.

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National Bureau of Standards
New York, N. Y.

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x	$\overline{H_0(x)}$	$\overline{H_1(x)}$	$\overline{L_0(x)}$	$\overline{L_1(x)}$
0.0	0.000000	0.000000	0.000000	0.000000
.1	.003181	.000071	.003185	.000071
.2	.012704	.000565	.012761	.000567
.3	.028505	.001903	.028791	.001917
.4	.050479	.004498	.051384	.004556
0.5	0.078480	0.008754	0.080690	0.008931
.6	.112322	.015060	.116906	.015501
.7	.151781	.023791	.160274	.024743
.8	.196597	.035301	.211086	.037156
.9	.246476	.049923	.269687	.053265
1.0	0.301090	0.067963	0.336473	0.073623
.1	.360084	.089703	.411897	.098820
.2	.423074	.115394	.496475	.129482
.3	.489655	.145255	.590787	.166281
.4	.559399	.179475	.695481	.209939
1.5	0.631863	0.218206	0.811283	0.261233
.6	.706590	.261566	0.938996	.321001
.7	.783111	.309637	1.079514	.390153
.8	.860954	.362463	1.233823	.469674
.9	0.939643	.420054	1.403013	.560633
2.0	1.018701	0.482381	1.588285	0.664194
.1	1.097659	.549379	1.790962	.781626
.2	1.176053	.620950	2.012499	0.914309
.3	1.253434	.696959	2.254495	1.063753
.4	1.329364	.777239	2.518708	1.231604
2.5	1.403427	0.861592	2.807067	1.419662
.6	1.475227	0.949789	3.121689	1.629896
.7	1.544392	1.041576	3.464899	1.864458
.8	1.610577	1.136671	3.839245	2.125706
.9	1.673465	1.234771	4.247522	2.416220
3.0	1.732773	1.335553	4.692798	2.738826
.1	1.788248	1.438677	5.178432	3.096622
.2	1.839675	1.543788	5.708112	3.493002
.3	1.886873	1.650520	6.285879	3.931686
.4	1.929699	1.758499	6.916164	4.416753
3.5	1.968046	1.867348	7.603826	4.952677
.6	2.001847	1.976687	8.354192	5.544365
.7	2.031071	2.086137	9.173103	6.197200
.8	2.055726	2.195325	10.066965	6.917087
.9	2.075858	2.303886	11.042802	7.710507
4.0	2.091545	2.411465	12.108317	8.584571
.1	2.102905	2.517720	13.271959	9.547086
.2	2.110084	2.622329	14.542995	10.606621
.3	2.113265	2.724986	15.931590	11.772582
.4	2.112655	2.825407	17.448895	13.055296
4.5	2.108492	2.923332	19.107144	14.466104
.6	2.101037	3.018528	20.919758	16.017458
.7	2.090574	3.110787	22.901465	17.723034
.8	2.077406	3.199932	25.068427	19.597850
.9	2.061852	3.285814	27.438379	21.658405
5.0	2.044244	3.368316	30.030788	23.922818

x	$\overline{H_0(x)}$	$\overline{H_1(x)}$	$\overline{L_0(x)}$	$\overline{L_1(x)}$
5.0	2.04424	3.36832	30.0308	23.9228
.1	2.02492	3.44735	32.8670	26.4110
.2	2.00424	3.52287	35.9705	29.1448
.3	1.98255	3.59485	39.3671	32.1483
.4	1.96021	3.66330	43.0849	35.4478
5.5	1.93756	3.72826	47.1550	39.0725
.6	1.91495	3.78980	51.6115	43.0540
.7	1.89272	3.84803	56.4919	47.4276
.8	1.87120	3.90308	61.8371	52.2317
.9	1.85069	3.95510	67.6925	57.5085
6.0	1.83148	4.00427	74.1074	63.3047
.1	1.81386	4.05081	81.1364	69.6714
.2	1.79808	4.09493	88.8392	76.6647
.3	1.78436	4.13687	97.2816	84.3465
.4	1.77291	4.17691	106.536	92.7847
6.5	1.76391	4.21531	116.681	102.054
.6	1.75750	4.25236	127.805	112.236
.7	1.75381	4.28835	140.002	123.422
.8	1.75292	4.32358	153.379	135.711
.9	1.75491	4.35834	168.051	149.212
7.0	1.75980	4.39296	184.144	164.044
.1	1.76758	4.42770	201.799	180.341
.2	1.77824	4.46288	221.170	198.247
.3	1.79171	4.49878	242.425	217.922
.4	1.80792	4.53566	265.749	239.542
7.5	1.82674	4.57378	291.347	263.300
.6	1.84804	4.61340	319.443	289.408
.7	1.87166	4.65473	350.283	318.101
.8	1.89743	4.69798	384.138	349.636
.9	1.92512	4.74334	421.307	384.295
8.0	1.95454	4.79097	462.117	422.390
.1	1.98544	4.84101	506.930	464.264
.2	2.01758	4.89356	556.140	510.293
.3	2.05070	4.94873	610.185	560.893
.4	2.08453	5.00656	669.543	616.519
8.5	2.11882	5.06710	734.742	677.675
.6	2.15328	5.13035	806.363	744.911
.7	2.18764	5.19629	885.042	818.837
.8	2.22165	5.26488	971.483	900.122
.9	2.25503	5.33605	1066.46	989.501
9.0	2.28754	5.40970	1170.81	1087.79
.1	2.31893	5.48572	1285.49	1195.87
.2	2.34897	5.56396	1411.51	1314.73
.3	2.37745	5.64427	1550.00	1445.46
.4	2.40416	5.72646	1702.22	1589.23
9.5	2.42894	5.81035	1869.52	1747.37
.6	2.45162	5.89572	2053.42	1921.30
.7	2.47207	5.98235	2255.58	2112.62
.8	2.49016	6.07001	2477.82	2323.08
.9	2.50581	6.15846	2722.15	2554.60
10.0	2.51896	6.24745	2990.77	2809.29

ON THE EXPLICIT NUMERICAL SOLUTION OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

By G. E. Hudson

Introduction. In standard texts on ordinary differential equations it is shown how to obtain the general solution y of the n^{th} order linear homogeneous differential equation with constant coefficients*,

$$y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' + b_n y = 0, \tag{1}$$

in which y is to be chosen to satisfy the initial conditions

$$y(0) = y_0\,, \quad y'(0) = y_0'\,, \cdots, y^{(n-1)}(0) = y_0^{(n-1)}\,. \tag{2}$$

It is shown that the general solution is of the form

$$y(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i-1} A_{ij}(x^j/j!) \exp\,(\alpha_i x) \tag{3}$$

with $\sum_{i=1}^{l} m_i = n$ where α_i is a root of multiplicity m_i of the secular equation

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0, \tag{4}$$

and l is the number of different roots.

The n arbitrary constants A_{ij} are to be determined by substituting (3) into the initial conditions (2), and by solving the resulting simultaneous linear algebraic equations.

Numerical methods for obtaining the roots of (4) are of course well known; explicit analytical formulas for the A_{ij} in terms of the b's and the initial values are rarely seen, whereas a really practical numerical procedure for calculating the A's other than for special cases, as far as the writer is aware, has not been published. Indeed, one assumes in the general case that, after finding the roots of the secular equation, one must then proceed, after differentiating (3) successively and substituting in the initial values, to solve n simultaneous linear algebraic equations for the A's by usual matrix or determinantal methods, or their equivalent[1]. It is the purpose of this paper to present the development of a numerical procedure which avoids such tedious methods of obtaining the A's.

Formal Development of the Method. For convenience, and because the initial conditions are introduced naturally thereby, the differential equation (1) is

* In this paper, the notation y' or $y^{(i)}$ means that y is differentiated once or i times, respectively, with respect to the independent variable x.

[1] J. S. Frame, "On the Explicit Solution of Simultaneous Linear Differential Equations with Constant Coefficients", Amer. Math. Mo. Vol. XLVII N. 1. (Jan., 1940).

first transformed into an integral equation by integrating it successively n times between the limits 0 and x. There results

$$y + b_1 \int_0^x y\,dx + b_2 \int_0^x \int_0^{x'} y\,dx''\,dx' + \cdots$$

$$+ b_n \int_0^x \cdots (n) \cdots \int_0^{x'} y\,dx'' \cdots (n) \cdots dx' \tag{5}$$

$$= c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}/(n-1)!$$

where the c's are constants of integration. It is easily verifiable that these constants are related to the initial values by the equations

$$c_0 = y_0\,, \quad c_1 = y_0' + b_1 y_0\,, \quad c_2 = y_0'' + b_1 y_0' + b_2 y_0\,, \cdots, \tag{6}$$

$$c_{n-1} = y_0^{(n-1)} + b_1 y_0^{(n-2)} + \cdots + b_{n-1} y_0\,,$$

so that, given the initial values, the c's can be calculated and vice versa. For some problems, the c's may be given whereas for some other problems the initial values may be given.

Upon taking the Laplace transform of equation (5), and of the expression (3), and combining them, there results

$$f(\lambda) \sum_{i=1}^{l} \sum_{j=0}^{m_i-1} A_{ij}/(\lambda - \alpha_i)^{j+1} = g(\lambda) \tag{7}$$

where λ is a parameter introduced by the transformation, and where $f(\lambda)$ and $g(\lambda)$ are defined by

$$f(\lambda) \equiv \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \equiv (\lambda - \alpha_1)^{m_1} (\lambda - \alpha_2)^{m_2} \cdots (\lambda - \alpha_l)^{m_l} \tag{8}$$

$$g(\lambda) \equiv c_0 \lambda^{n-1} + c_1 \lambda^{n-2} + \cdots + c_{n-1} \tag{9}$$

(Equation (7) forms the basis for the numerical calculation of the coefficients A_{ij}.

Now define the auxiliary polynomials

$$f_{ij}(\lambda) = \frac{f(\lambda)}{(\lambda - \alpha_i)^{j+1}} \qquad j = 0, \cdots, m_i - 1 \qquad i = 1, \cdots, l \tag{10}$$

which therefore have the properties that

$$f_{ij}^{(k)}(\alpha_i) = 0 \qquad k = 0, 1, \cdots, m_i - j - 2\,, \qquad j = 0, 1, \cdots, m_i - 2 \tag{11'}$$

and

$$f_{pj}^{(k)}(\alpha_i) = 0, i \neq p, k = 0, 1, \cdots, m_i - 1\,, \quad j = 0, 1, \cdots, m_p - 1 \tag{11''}$$

Let us concentrate on finding the set of coefficients $A_{i0}, A_{i1}, \cdots, A_{i(m_i-1)}$ for a fixed value of i. If we differentiate equation (7) k times, where $0 \leq k \leq m_i - 1$, put $\lambda = \alpha_i$, and divide by $k!$, then, in view of equations (11) we find

$$\sum_{j=m_i-1-k}^{m_i-1} f_{ij}^{(k)}(\alpha_i)/k!\, A_{ij} = g^{(k)}(\alpha_i)/k!, \qquad (k = 0, 1, \cdots, m_i - 1) \tag{12}$$

and there is one such set of equations in "canonical" form for each value of i from 1 to l. From (9) and (6), it is clear that

$$g^{(k)}(\alpha_i) = [(d^k/d\lambda^k)\ \{y_0(\lambda^{n-1} + b_1\lambda^{n-2} + \cdots + b_{n-1}) + y_0'(\lambda^{n-2} + b_1\lambda^{n-3} + \cdots + b_{n-2}) + \cdots + y_0^{(n-2)}(\lambda + b_1) + y_0^{(n-1)}\}]_{\lambda=\alpha_i}$$

where $k = 0, \cdots m_i - 1$. Hence the coefficients of the initial values in $g^{(k)}(\alpha_i)/k!$ appear as the coefficients of $f_{ik}(\lambda)$, in the $(k+1)$st quotient row in the schema for the division of $f(\lambda)$ by $(\lambda - \alpha_i)$ m_i times, using Newton's method of synthetic division.

If, now, the division by $\lambda - \alpha_i$ is continued another m_i times using Newton's method, the remainder terms in the resulting quotient rows are related to the coefficients $f_{ij}^{(k)}(\alpha_i)/k!$ in (12), which are being sought. Indeed the remainder term in the $(m_i + 1)$st quotient row is

$$f_{i(m_i-1)}(\alpha_i) = f'_{i(m_i-2)}(\alpha_i) = \cdots = f_{i0}^{(m_i-1)}(\alpha_i)/(m_i - 1)!$$

that in the $(m_i + 2)$nd quotient row is

$$f'_{i(m_i-1)}(\alpha_i) = f''_{i(m_i-2)}(\alpha_i)/2! = \cdots = f_{i1}^{(m_i-1)}(\alpha_i)/(m_i - 1)!$$

and so forth, up to the remainder term in the $(2m_i)$th quotient row, which is

$$f_{i(m_i-1)}^{(m_i-1)}(\alpha_i)/(m_i - 1)!$$

Thus it appears that Newton's method of division is peculiarly suited for calculating the various coefficients and numbers needed in Equation (12), from which in turn the values of the A_{ij} can be easily obtained.

Summary of the Numerical Procedure. Given the differential equation (1) to be solved subject to the initial conditions (2), first calculate the roots α_i of the secular equation (4). Then, take any root, say α_i, of multiplicity m_i, and divide the polynomial $f(\lambda)$, defined in (8), by the factor $\lambda - \alpha_i$ in succession $2m_i$ times by Newton's method of synthetic division. Using the numbers appearing in this schema, write down the equations (12) for the constants A_{ij} appearing in the general solution (3). To do this, we note that (a) the coefficient of A_{ij} in (12), namely,

$$f_{ij}^{(k)}(\alpha_i)/k!$$

appears as the remainder term in the $(j + k + 2)$nd quotient row in the schema, and (b) the coefficient of $y_0^{(r-1)}$ in the non-homogeneous term $g^{(k)}(\alpha_i)/k!$ in (12) appears as the rth number, counting from right to left (not counting the zero remainder term), in the $(k+1)$st quotient row. Finally, we solve the equations (12) for the A_{ij}'s, beginning with the one involving only $A_{i(m_i-1)}$, proceeding to the next involving only $A_{i(m_i-2)}$ and $A_{i(m_i-1)}$, and so forth.

Acknowledgement. The writer expresses his thanks to Dr. B. Miller for pointing out the usefulness of such an explicit numerical procedure in connection with the problems which occur, for instance, in vibration analysis in engineering, and to Mr. J. Neuringer for checking the algebra.

Upon request the writer will be pleased to furnish a sample of the application of this method to a numerical example, computed by Mrs. E. L. Miller.

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A SHORT TABLE OF STRUVE FUNCTIONS AND OF SOME INTEGRALS INVOLVING BESSEL AND STRUVE FUNCTIONS*

By C. W. Horton

The Struve function $H_n(z)$ defined by Watson[1] is tabulated in Table I for $n = 2(1)4$, $z = 0(0.1)10.0$ to $4D$. The values have been calculated to $7D$ by two different processes. The final values have been rounded off to $4D$ because of the coarseness of the interval. The values are calculated by means of the infinite series for $H_n(z)$ and by means of the well known recurrence formula

$$H_{n+1}(z) = (2n/z)H_n(z) - H_{n-1}(z) + (z/2)^n/\sqrt{\pi}\Gamma(n + \tfrac{3}{2}).$$

The values of $H_0(z)$ and $H_1(z)$ calculated by Watson[2] were used as a starting point.

The integral $C_n(z) = \int_0^z z^n J_n(z)\,dz$ is tabulated in Table I for $n = 1(1)4$, $z = 0(0.1)10.0$ to $4D$. The values have been calculated by numerical integration and by means of the recurrence formula

$$C_{n+1}(z) = (2n + 1)C_n(z) - z^{n+1}J_n(z).$$

Values of $C_0(z)$ were obtained from a table by Lowan and Abramowitz[3]. For $n = 3$ and 4 and z between 6.0 and 10.0 the differences in $C_n(z)$ become so large that the values are tabulated to $3D$ only.

The integral $D_n(z) = \int_0^z z^n H_n(z)\,dz$ is tabulated in Table I for $n = 0(1)4$, $z = 0(0.1)10.0$ to $4D$. The values of $D_0(z)$ were obtained from a manuscript table $0(0.1)10.0$ to $8D$ which was generously given to the writer by Dr. J. W. Wrench, Jr.[4] The remaining values would probably not have been calculated, if this table had not been available. The values of $D_n(z)$, $n \neq 0$, were calculated by means of the recurrence formula

$$D_{n+1}(z) = (2n + 1)D_n(z) - z^{n+1}H_n(z) + z^{2n+2}/(n + 1)2^{n+1}\Gamma(\tfrac{1}{2})\Gamma(n + \tfrac{3}{2}).$$

These calculations were checked by numerical integration. For $n = 3$ and 4 and z between 6.0 and 10.0 the differences in $D_n(z)$ become so large that the values are tabulated to $3D$ only.

Since every tabulated value has been calculated by two independent methods, it is believed that the maximum error in any value is 0.6 of the last decimal.

The University of Texas.

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[1] Watson, G. N., Bessel Functions p. 328, eq. (1), The Macmillan Co., New York, 1945.

[2] *Ibid.*, pp. 666–685.

[3] Lowan, A. N., and Abramowitz, M., J. Math. Phys., **22**, 3–12, 1943.

[4] Wrench, John W., Jr., Math. Tab. and Aids Comp. III, 66, 1948.

TABLE I

z	$H_3(z)$	$H_2(z)$	$H_4(z)$	$C_1(z)$	$C_2(z)$	$C_3(z)$	$C_4(z)$	$D_0(z)$	$D_1(z)$	$D_2(z)$	$D_3(z)$	$D_4(z)$	z
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
0.1	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0032	0.0000	0.0000	0.0000	0.0000	0.1
0.2	0.0003	0.0000	0.0000	0.0013	0.0000	0.0000	0.0000	0.0127	0.0001	0.0000	0.0000	0.0000	0.2
0.3	0.0011	0.0000	0.0000	0.0045	0.0001	0.0000	0.0000	0.0285	0.0004	0.0000	0.0000	0.0000	0.3
0.4	0.0027	0.0002	0.0000	0.0105	0.0003	0.0000	0.0000	0.0505	0.0013	0.0000	0.0000	0.0000	0.4
0.5	0.0052	0.0004	0.0000	0.0204	0.0008	0.0000	0.0000	0.0785	0.0033	0.0001	0.0000	0.0000	0.5
0.6	0.0090	0.0008	0.0001	0.0350	0.0019	0.0001	0.0000	0.1123	0.0068	0.0003	0.0000	0.0000	0.6
0.7	0.0142	0.0014	0.0001	0.0551	0.0041	0.0002	0.0000	0.1518	0.0125	0.0008	0.0000	0.0000	0.7
0.8	0.0211	0.0024	0.0002	0.0813	0.0079	0.0006	0.0000	0.1966	0.0211	0.0018	0.0001	0.0000	0.8
0.9	0.0298	0.0039	0.0004	0.1143	0.0141	0.0014	0.0001	0.2465	0.0336	0.0037	0.0003	0.0000	0.9
1.0	0.0405	0.0058	0.0007	0.1545	0.0235	0.0028	0.0003	0.3011	0.0507	0.0068	0.0007	0.0001	1.0
1.1	0.0533	0.0085	0.0010	0.2024	0.0374	0.0055	0.0007	0.3601	0.0736	0.0120	0.0016	0.0002	1.1
1.2	0.0685	0.0119	0.0016	0.2583	0.0571	0.0099	0.0014	0.4231	0.1032	0.0201	0.0031	0.0004	1.2
1.3	0.0860	0.0163	0.0024	0.3220	0.0839	0.0172	0.0029	0.4897	0.1405	0.0321	0.0059	0.0009	1.3
1.4	0.1061	0.0217	0.0034	0.3939	0.1195	0.0285	0.0055	0.5594	0.1868	0.0497	0.0106	0.0019	1.4
1.5	0.1286	0.0282	0.0048	0.4737	0.1658	0.0455	0.0101	0.6319	0.2430	0.0744	0.0182	0.0037	1.5
1.6	0.1538	0.0361	0.0065	0.5612	0.2246	0.0705	0.0179	0.7066	0.3102	0.1083	0.0302	0.0070	1.6
1.7	0.1816	0.0455	0.0088	0.6559	0.2980	0.1059	0.0305	0.7831	0.3896	0.1540	0.0485	0.0126	1.7
1.8	0.2120	0.0564	0.0115	0.7574	0.3882	0.1553	0.0502	0.8610	0.4821	0.2143	0.0759	0.0222	1.8
1.9	0.2449	0.0691	0.0150	0.8650	0.4971	0.2227	0.0804	0.9396	0.5886	0.2925	0.1157	0.0377	1.9
2.0	0.2803	0.0836	0.0191	0.9780	0.6271	0.3127	0.1256	1.0187	0.7102	0.3924	0.1724	0.0624	2.0
2.1	0.3182	0.1001	0.0241	1.0954	0.7801	0.4310	0.1915	1.0977	0.8476	0.5183	0.2516	0.1006	2.1
2.2	0.3583	0.1187	0.0300	1.2163	0.9581	0.5841	0.2858	1.1761	1.0015	0.6747	0.3604	0.1584	2.2
2.3	0.4006	0.1394	0.0369	1.3397	1.1631	0.7793	0.4183	1.2534	1.1726	0.8669	0.5076	0.2443	2.3
2.4	0.4449	0.1625	0.0451	1.4643	1.3966	1.0249	0.6015	1.3294	1.3613	1.1005	0.7036	0.3695	2.4
2.5	0.4911	0.1878	0.0545	1.5889	1.6600	1.3302	0.8507	1.4034	1.5680	1.3815	0.9614	0.5490	2.5
2.6	0.5389	0.2156	0.0652	1.7124	1.9544	1.7053	1.1849	1.4752	1.7929	1.7166	1.2962	0.8023	2.6
2.7	0.5882	0.2459	0.0775	1.8333	2.2807	2.1611	1.6370	1.5444	2.0362	2.1125	1.7259	1.1548	2.7
2.8	0.6387	0.2786	0.0915	1.9504	2.6391	2.7094	2.2045	1.6106	2.2977	2.5766	2.2716	1.6386	2.8
2.9	0.6902	0.3139	0.1072	2.0623	3.0296	3.3626	2.9500	1.6735	2.5773	3.1165	2.9579	2.2946	2.9
3.0	0.7424	0.3517	0.1247	2.1677	3.4516	4.1337	3.9020	1.7328	2.8746	3.7401	3.8128	3.1735	3.0
3.1	0.7951	0.3920	0.1443	2.2653	3.9041	5.0361	5.1049	1.7882	3.1892	4.4556	4.8685	4.3386	3.1
3.2	0.8481	0.4348	0.1659	2.3539	4.3855	6.0833	6.6099	1.8397	3.5203	5.2711	6.1615	5.8671	3.2
3.3	0.9010	0.4801	0.1898	2.4322	4.8936	7.2889	8.4752	1.8869	3.8672	6.1952	7.7328	7.8532	3.3
3.4	0.9536	0.5277	0.2160	2.4992	5.4256	8.6663	10.7666	1.9297	4.2289	7.2363	9.6283	10.4104	3.4
3.5	1.0057	0.5777	0.2446	2.5538	5.9784	10.2282	13.5577	1.9680	4.6045	8.4027	11.8990	13.6751	3.5
3.6	1.0569	0.6298	0.2756	2.5950	6.5479	11.9867	16.9298	2.0018	4.9926	9.7029	14.6014	17.8090	3.6
3.7	1.1071	0.6840	0.3093	2.6222	7.1297	13.9525	20.9723	2.0311	5.3921	11.1449	17.7974	23.0036	3.7
3.8	1.1559	0.7402	0.3456	2.6347	7.7189	16.1352	25.7824	2.0557	5.8016	12.7366	21.5546	29.4833	3.8
3.9	1.2032	0.7983	0.3846	2.6318	8.3099	18.5423	31.4648	2.0759	6.2195	14.4855	25.9485	37.5098	3.9
4.0	1.2487	0.8580	0.4264	2.6133	8.8967	21.1793	38.1310	2.0915	6.6444	16.3989	31.0527	47.3871	4.0
4.1	1.2922	0.9193	0.4709	2.5790	9.4729	24.0490	45.8988	2.1029	7.0748	18.4834	36.9586	59.4652	4.1
4.2	1.3335	0.9819	0.5183	2.5287	10.0317	27.1517	54.8915	2.1101	7.5089	20.7452	43.7555	74.1455	4.2
4.3	1.3725	1.0456	0.5686	2.4626	10.5661	30.4841	65.2360	2.1133	7.9452	23.1898	51.5408	91.8863	4.3
4.4	1.4090	1.1104	0.6217	2.3810	11.0686	34.0397	77.0622	2.1127	8.3830	25.8222	60.4179	113.2074	4.4
4.5	1.4428	1.1759	0.6776	2.2843	11.5319	37.8079	90.5004	2.1085	8.8177	28.6466	70.4957	138.6967	4.5
4.6	1.4739	1.2421	0.7364	2.1732	11.9484	41.7743	105.6799	2.1010	9.2506	31.6666	81.8886	169.0145	4.6
4.7	1.5021	1.3086	0.7980	2.0486	12.3106	45.9201	122.7262	2.0906	9.6798	34.8849	94.7166	204.9010	4.7
4.8	1.5275	1.3754	0.8623	1.9113	12.6112	50.2217	141.7590	2.0774	10.1032	38.3035	109.1045	247.1808	4.8
4.9	1.5499	1.4422	0.9293	1.7624	12.8430	54.6513	162.8887	2.0619	10.5197	41.9238	125.1821	296.7698	4.9
5.0	1.5694	1.5087	0.9990	1.6033	12.9994	59.1762	186.2138	2.0442	10.9281	45.7462	143.0835	354.6810	5.0
5.1	1.5859	1.5749	1.0712	1.4353	13.0738	63.7588	211.8175	2.0249	11.3272	49.7705	162.9469	422.0310	5.1
5.2	1.5996	1.6406	1.1459	1.2600	13.0607	68.3572	239.7644	2.0042	11.7161	53.9957	184.9143	500.0452	5.2
5.3	1.6104	1.7055	1.2230	1.0789	12.9548	72.9247	270.0964	1.9826	12.0939	58.4204	209.1311	590.0647	5.3
5.4	1.6185	1.7695	1.3024	0.8938	12.7518	77.4104	302.8298	1.9602	12.4601	63.0421	235.7456	693.5319	5.4

TABLE I—*Continued*

z	$H_2(z)$	$H_3(z)$	$H_4(z)$	$C_1(z)$	$C_2(z)$	$C_3(z)$	$C_4(z)$	$D_0(z)$	$D_1(z)$	$D_2(z)$	$D_3(z)$	$D_4(z)$	z
5.5	1.6239	1.8325	1.3839	0.7066	12.4482	81.7502	337.9508	1.9376	12.8141	67.8584	264.9088	812.0064	5.5
5.6	1.6269	1.8944	1.4675	0.5189	12.0413	85.9126	375.4123	1.9150	13.1557	72.8659	296.7740	947.4211	5.6
5.7	1.6275	1.9549	1.5531	0.3327	11.5298	89.8087	415.1302	1.8927	13.4846	78.0611	331.4966	1101.3879	5.7
5.8	1.6260	2.0140	1.6405	0.1500	10.9130	93.3830	456.9797	1.8712	13.8011	83.4402	369.2332	1276.0032	5.8
5.9	1.6225	2.0717	1.7296	−0.0274	10.1916	96.5691	500.7924	1.8507	14.1054	88.9994	410.1425	1473.4234	5.9
6.0	1.6172	2.1278	1.8203	−0.1977	9.3677	99.2989	546.3528	1.8315	14.3980	94.7345	454.3841	1695.9608	6.0
6.1	1.6104	2.1824	1.9124	−0.3588	8.4442	101.504	593.395	1.8139	14.6795	100.6419	502.119	1946.089	6.1
6.2	1.6022	2.2352	2.0059	−0.5092	7.4256	103.117	641.607	1.7981	14.9508	106.7177	553.509	2226.448	6.2
6.3	1.5931	2.2865	2.1006	−0.6471	6.3176	104.070	690.615	1.7844	15.2129	112.9589	608.718	2539.850	6.3
6.4	1.5831	2.3361	2.1964	−0.7709	5.1272	104.298	739.997	1.7729	15.4672	119.3626	667.910	2889.287	6.4
6.5	1.5725	2.3841	2.2932	−0.8791	3.8624	103.740	789.376	1.7639	15.7148	125.9269	731.253	3277.933	6.5
6.6	1.5617	2.4306	2.3910	−0.9705	2.5327	102.338	837.919	1.7575	15.9575	132.6504	798.914	3709.153	6.6
6.7	1.5509	2.4755	2.4895	−1.0437	1.1487	100.040	885.343	1.7538	16.1968	139.5329	871.066	4186.510	6.7
6.8	1.5404	2.5191	2.5888	−1.0979	−0.2781	96.800	930.911	1.7529	16.4346	146.5751	947.882	4713.769	6.8
6.9	1.5303	2.5614	2.6887	−1.1322	−1.7349	92.578	973.944	1.7549	16.6728	153.7790	1029.542	5294.907	6.9
7.0	1.5210	2.6025	2.7893	−1.1459	−3.2083	87.345	1013.714	1.7598	16.9133	161.1479	1116.230	5934.123	7.0
7.1	1.5127	2.6425	2.8904	−1.1386	−4.6839	81.078	1049.457	1.7676	17.1583	168.6867	1208.136	6635.845	7.1
7.2	1.5057	2.6818	2.9921	−1.1102	−6.1468	73.768	1080.381	1.7782	17.4098	176.4016	1305.457	7404.739	7.2
7.3	1.5001	2.7203	3.0943	−1.0604	−7.5815	65.413	1105.663	1.7917	17.6701	184.3007	1408.398	8245.724	7.3
7.4	1.4962	2.7583	3.1971	−0.9897	−8.9722	56.027	1124.466	1.8079	17.9411	192.3935	1517.176	9163.963	7.4
7.5	1.4942	2.7959	3.3004	−0.8984	−10.3029	45.632	1135.947	1.8267	18.2252	200.6917	1632.019	10164.977	7.5
7.6	1.4943	2.8335	3.4042	−0.7871	−11.5574	34.268	1139.264	1.8480	18.5243	209.2084	1753.166	11254.463	7.6
7.7	1.4965	2.8712	3.5087	−0.6567	−12.7201	21.983	1133.589	1.8717	18.8405	217.9588	1880.874	12438.507	7.7
7.8	1.5011	2.9092	3.6139	−0.5083	−13.7753	8.845	1118.120	1.8974	19.1757	226.9600	2015.416	13723.511	7.8
7.9	1.5081	2.9477	3.7199	−0.3430	−14.7081	−5.070	1092.063	1.9251	19.5318	236.2307	2157.083	15110.229	7.9
8.0	1.5177	2.9870	3.8268	−0.1625	−15.5041	−19.669	1054.798	1.9545	19.9105	245.7915	2306.188	16623.794	8.0
8.1	1.5299	3.0273	3.9347	+0.0318	−16.1501	−34.845	1005.581	1.9854	20.3133	255.6649	2463.065	18253.741	8.1
8.2	1.5447	3.0688	4.0437	0.2380	−16.6337	−50.478	943.878	2.0176	20.7416	265.8749	2628.075	20014.040	8.2
8.3	1.5622	3.1117	4.1540	0.4543	−16.9439	−66.438	869.212	2.0507	21.1968	276.4470	2801.605	21913.127	8.3
8.4	1.5824	3.1564	4.2657	0.6785	−17.0713	−82.580	781.214	2.0845	21.6797	287.4085	2984.070	23959.937	8.4
8.5	1.6052	3.2028	4.3791	0.9085	−17.0077	−98.748	679.634	2.1188	22.1913	298.7876	3175.916	26163.941	8.5
8.6	1.6306	3.2514	4.4942	1.1420	−16.7469	−114.779	564.357	2.1533	22.7321	310.6140	3377.623	28535.187	8.6
8.7	1.6586	3.3022	4.6113	1.3768	−16.2846	−130.499	435.415	2.1876	23.3025	322.9183	3589.705	31084.341	8.7
8.8	1.6889	3.3554	4.7307	1.6105	−15.6182	−145.728	292.998	2.2216	23.9027	335.7318	3812.712	33822.735	8.8
8.9	1.7215	3.4112	4.8524	1.8408	−14.7476	−160.280	137.465	2.2550	24.5326	349.0867	4047.231	36762.416	8.9
9.0	1.7563	3.4698	4.9768	2.0653	−13.6744	−173.966	−30.646	2.2875	25.1918	363.0152	4293.890	39916.194	9.0
9.1	1.7931	3.5312	5.1041	2.2816	−12.4028	−186.596	−210.612	2.3189	25.8797	377.5500	4553.356	43297.606	9.1
9.2	1.8317	3.5956	5.2344	2.4876	−10.9388	−197.981	−401.518	2.3490	26.5956	392.7237	4826.338	46921.432	9.2
9.3	1.8720	3.6631	5.3681	2.6809	−9.3910	−207.934	−602.251	2.3774	27.3385	408.5686	5113.588	50802.838	9.3
9.4	1.9137	3.7337	5.5054	2.8596	−7.4701	−216.272	−811.600	2.4042	28.1070	425.1165	5415.899	54956.352	9.4
9.5	1.9567	3.8075	5.6464	3.0218	−5.4888	−222.822	−1027.757	2.4289	28.8998	442.3983	5734.108	59405.489	9.5
9.6	2.0006	3.8846	5.7915	3.1655	−3.3623	−227.419	−1249.318	2.4516	29.7151	460.4441	6069.097	64163.811	9.6
9.7	2.0453	3.9649	5.9408	3.2891	−1.1071	−229.911	−1474.290	2.4721	30.5511	479.2827	6421.790	69250.194	9.7
9.8	2.0906	4.0484	6.0946	3.3913	+1.2575	−230.161	−1700.598	2.4902	31.4058	498.9415	6793.153	74688.697	9.8
9.9	2.1361	4.1351	6.2530	3.4707	3.7112	−228.048	−1925.909	2.5058	32.2770	519.4462	7184.196	80500.729	9.9
10.0	2.1817	4.2250	6.4163	3.5264	6.2318	−223.471	−2148.002	2.5190	33.1625	540.8208	7595.968	86710.097	10.0

ON SOME GENERAL INTEGRATION METHODS EMPLOYED IN CONNECTION WITH LINEAR DIFFERENTIAL EQUATIONS

By Gaetano Fichera.[1]

In a recent paper,[2] J. B. Diaz and H. J. Greenberg have studied the first biharmonic boundary value problem. This paper prompted the following exposition of some general analytical methods for the solution of physical problems, that have been developed by M. Picone and his co-workers.

I shall consider the application of these methods to the particular case of the biharmonic problem.

Let R be a domain with boundary C in the xy plane, and $w(x, y)$ a solution of the following biharmonic boundary value problem

$$\begin{aligned} &\Delta\Delta w = p \text{ in } R, \\ &w = f, \qquad dw/dn = g \text{ on } C, \end{aligned} \tag{1}$$

where n is the outer normal to C, $p(x, y)$ is a given function of x and y in the domain R, $f(s)$ and $g(s)$ are given functions of the variable s along the arc C.

In N a procedure is outlined for obtaining from convenient complete sets of functions a sequence $\{b_h(x, y)\}$ satisfying at the point (x_0, y_0) the condition

$$\lim_{h\to\infty} b_h(x_0, y_0) = w(x_0, y_0).$$

To this end the authors employ two sets of functions $\{u_i\}$ and $\{v_i\}$ having the following properties

$$u_i = du_i/dn = 0 \text{ on } C, \qquad \Delta\Delta v_i = 0 \text{ in } R$$

$$\int_R \Delta u_i \, \Delta u_j \, dx\, dy \begin{cases} = 0 & \text{if } i \neq j. \\ = 1 & \text{if } i = j; \end{cases}$$

$$\int_R \Delta v_i \, \Delta v_j \, dx\, dy \begin{cases} = 0 & \text{if } i \neq j, \\ = 1 & \text{if } i = j. \end{cases}$$

Moreover the following conditions must be satisfied by $\{u_i\}$ and $\{v_i\}$: if u is such that $u = du/dn = 0$ on C and $(\Delta u)^2$ is integrable in R, then

$$\sum_{i=1}^{\infty} \left\{ \int_R \Delta u \cdot \Delta u_i \, dx\, dy \right\}^2 = \int_R (\Delta u)^2 \, dx\, dy;$$

if v is such that $\Delta\Delta v = 0$ in R, then

$$\sum_{i=1}^{\infty} \left\{ \int_R \Delta v \cdot \Delta v_i \, dx\, dy \right\}^2 = \int_R (\Delta v)^2 \, dx\, dy.$$

[1] This paper has been prepared at the National Institute for the Applications of the Calculus, Rome, Italy.

[2] "Upper and lower bounds of the first biharmonic boundary value problem," J. Math. and Phys., 27 (1948) pp. 193–217. This paper shall be referred to as N.

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In addition, the authors consider four functions u, v, $\bar{u}$, $\bar{v}$ defined in R and satisfying the following conditions:

$$u = w = f, \qquad \frac{du}{dn} = \frac{dw}{dn} = g \qquad \text{on} \quad C,$$

$$\Delta\Delta v = \Delta\Delta w = p \qquad \text{in } R,$$

$$\bar{u} = -r^2 \log r, \qquad \frac{d\bar{u}}{dn} = -\frac{d}{dn} r^2 \log r \qquad \text{on} \quad C,$$

$$\Delta\Delta\bar{v} = 0 \qquad \text{in } R$$

where r is the distance of an arbitrary point $P \equiv (x, y)$ from a fixed point $P_0 \equiv (x_0, y_0)$.

The sequences $\{b_h(P)\}$ can be simply obtained by the following procedure, founded on the above mentioned methods. We will assume that the same hypotheses hold, as in N, w and R being regular in order to make the application of Green's identity possible.

The function w can be expressed as the sum

$$w = w_0 + w_1$$

where

$$w_0 = w, \qquad dw_0/dn = dw/dn \text{ on } C, \qquad \Delta\Delta w_0 = 0 \text{ in } R,$$

$$w_1 = dw_1/dn = 0 \text{ on } C, \qquad \Delta\Delta w_1 = \Delta\Delta w \text{ in } R.$$

By Green's identity one gets

$$w(P_0) = \frac{1}{2\pi} \int_C \left(w \frac{d \log r}{dn} - \frac{dw}{dn} \log r \right) ds + \frac{1}{2\pi} \int_R \Delta w_0 \log r \, dx \, dy + \frac{1}{2\pi} \int_R \Delta w_1 \log r \, dx \, dy.$$

In order to obtain w it will then suffice to know the Fourier coefficients c_i of Δw_0 with respect to the system $\{\Delta v_i\}$ and the corresponding coefficients γ_i of Δw_1 with respect to $\{\Delta u_i\}$. The b_h are then given by

$$b_h(P_0) = \frac{1}{2\pi} \int_C \left(w \frac{d \log r}{dn} - \frac{dw}{dn} \log r \right) ds + \frac{1}{2\pi} \sum_{i=1}^{h} c_i \int_R \Delta v_i \log r \, dx \, dy + \frac{1}{2\pi} \sum_{i=1}^{h} \gamma_i \int_R \Delta u_i \log r \, dx \, dy.$$

Applying Green's lemma the coefficients c_i and γ_i are expressed as follows

$$c_i = \int_R \Delta w_0 \Delta v_i \, dx \, dy = \int_C \left(\Delta v_i \frac{dw_0}{dn} - w_0 \frac{d\Delta v_i}{dn} \right) ds,$$

$$\gamma_i = \int_R \Delta w_1 \Delta u_i \, dx \, dy = \int_R u_i \Delta\Delta w_1 \, dx \, dy,$$

where the right hand members are known from (3).

The above $b_h(P_0)$ converges uniformly to $w(P_0)$ throughout $R + C$, since:

$$[2\pi(w(P_0) - b_h(P_0))]^2 \leq \sum_{h+1}^{\infty} (c_i^2 + \gamma_i^2) \int_R (\log r)^2 \, dx \, dy.$$

The same is true for the first derivatives of $b_h(P_0)$, the convergence to the corresponding first derivatives of $w(P_0)$ being however limited to any closed set interior to R.

The last procedure, which does not employ the functions $u, v, \bar{u}, \bar{v}$, can be considered as a special case of a more general integration method for linear differential equations derived by M. Picone[3] from Green's identity and which has been used for some time in the Italian Institute for the Applications of the Calculus. This method, that we outline below in connection with the bi-harmonic problem, is easily adapted to numerical computations, and has reecived a complete theoretical justification.[4] It should be observed here that nothing is said in N about the existence of the complete sets of functions employed.

Problems of the above kind, related to the existence of complete sets of functions, have been recently solved from the standpoint of functional topology by M. Picone's co-workers.[5]

1. Three different methods will now be outlined for the solution of the bi-harmonic problem. If $p = \Delta\Delta w \neq 0$, the solution of problem (1) can be reduced

[3] See M. Picone: a) *Appunti di Analisi superiore*, Rondinella (Napoli) 1940, pp. 752–65; b) *Nuovi metodi risolutivi per i problemi d'integrazione delle equazioni lineari a derivate parziali e nuove applicazioni della trasformata multipla di Laplace nel caso delle equazioni a coefficienti costanti*, Atti dell'Acc. delle Scienze di Torino, vol. 75, 1939–40; c) *Sulla traduzione in equazione integrale lineare di prima specie dei problemi al contorno concernenti i sistemi di equazioni lineari a derivate parziali*, Rend. Acc. Naz. Lincei, serie VIII, vol. II, 1947.

[4] See L. Amerio: a) *Sull'integrazione dell'equazione* $\Delta_2 u - \lambda^2 u = f$ *in un dominio di connessione qualsiasi*, Rend. Istituto Lombardo di Scienze e Lettere, vol. 78, 1944–45; b) *Sull'integrazione dell'equazione* $\Delta_{2k} u = f$, Annali di Matematica, tomo 24, serie IV; c) *Sull'equazione di propagazione del calore*; Rend. di Mat. e delle sue Applicazioni, Roma, 1946; d) *Sul calcolo delle soluzioni dei problemi al contorno per le equazioni del secondo ordine di tipo ellittico*, American Journal of Mathematics, vol. LXIX, n. 3, July 1947; A. Ghizzetti: a) *Sul metodo della trasformata parziale di Laplace ad intervallo d'integrazione finito*, Rend. di Mat. e delle sue Applicazioni, Roma, 1947; b) *Applicazione del metodo della trasformata parziale di Laplace al problema di Dirichlet per l'equazione* $\Delta_2 u - \lambda^2 u = F$ *in n variabili*, Rend. Seminario Mat. Università di Padova, vol. XVII, 1948; G. Fichera: a) *Sull'integrazione delle equazioni dell'elasticità*, Rend. Acc. Naz. Lincei, serie VIII, vol. II, Fasc. 4, 1947; b) *Sull'equilibrio di un corpo elastico isotropo e omogeneo*, Rend. Seminario Mat. Univ. Padova, vol. XVII, 1948.

[5] See G. Fichera: a) *Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni*, Annali di Mat. pura ed applicata, serie IV, Tomo XXVII, Fasc. 1–2, 1948; b) *Teoremi di completezza connessi all'integrazione dell'equazione* $\Delta_4 u = f$, Giornale di Matematiche di Battaglini, s. IV, v. 37, 1947; c) *Sull'approssimazione delle funzioni armoniche in tre variabili mediante successioni di particolari funzioni armoniche*, Rend. Acc. Lincei, serie VIII, vol. III, Fasc. 5–6, 1947; d) *Applicazione della teoria del potenziale di superficie ed alcuni problemi di analisi funzionale lineare*, Giornale di Matematiche di Battaglini, s. IV, vol. 78, 1948–49.

to that of the following problem

$$w'' = w - w' \quad \text{on} \quad C, \ \Delta\Delta w'' = 0 \quad \text{in} \quad R$$

$$\frac{dw''}{dn} = \frac{dw}{dn} - \frac{dw'}{dn} \quad \text{on} \quad C,$$

where $w' = (1/2\pi) \int_R \Delta\Delta w \log r \, dx \, dy$.

We can therefore assume without any loss in generality that $p = \Delta\Delta w = 0$.

For R simply-connected[6] we let $\{\omega_k\}$ denote a set of homogeneous, linearly independent, bi-harmonic polynomials whose linear combinations give all by-harmonic polynomials.[7]

If $G(P, Q)$ denotes Green's function for problem (1), the solution is given by the well-known formula

$$(4) \qquad w(P) = \frac{1}{8\pi} \int_C \left\{ f(Q) \frac{d\Delta G(P,Q)}{dn} - g(Q) \Delta G(P,Q) \right\} d_Q s$$

If we approximate the vector (f, g) in the mean by means of the vectors of the set $(\omega_k , d\omega_k/dn)$ the h'th approximation is given by

$$f_h = \sum_{k=0}^{h} c_k^{(h)} \omega_k , \qquad g_h = \sum_{k=0}^{h} c_k^{(h)} \, d\omega_k/dn,$$

provided the $c_k^{(h)}$ make the following quadratic polynomials in γ_k a minimum

$$\int_C \left[f - \sum_{k=0}^{h} \gamma_k \omega_k \right]^2 ds + \int_C \left[g - \sum_{k=0}^{h} \gamma_k \frac{d\omega_k}{dn} \right]^2 ds,$$

this means that the $c_k^{(h)}$ are solutions of the following linear system

$$(5) \qquad \sum_{j=0}^{h} \gamma_j \int_C \left(\omega_k \, \omega_j + \frac{d\omega_k}{dn} \frac{d\omega_j}{dn} \right) ds = \int_C \left(f\omega_k + g \frac{d\omega_k}{dn} \right) ds.$$

The following completeness theorem can then be proved.[8]

The sequences f_h and g_h converge in the mean on C to f and g. That is

$$\lim_{h \to \infty} \int_C [f - f_h]^2 \, ds = \lim_{h \to \infty} \int_C [g - g_h]^2 \, ds = 0$$

and therefore:

[6] For the case of multiconnected R see G. Fichera, loc. cit. b) (5).

[7] A system of this kind can be obtained by taking for every integral value $k \geq 0$ the following polynomials

$$\mathcal{R}(x + iy)^k, \quad \mathcal{I}(x + iy)^k, \quad \mathcal{R}(x^2 + y^2)(x + iy)^k, \quad \mathcal{I}(x^2 + y^2)(x + iy)^k.$$

[8] See G. Fichera b) loc. cit. (5).

If R' is an arbitrary closed set interior to R, the sequence

$$w_h(x, y) = \sum_{k=1}^{h} c_k^{(h)} \omega_k(x, y)$$

converges uniformly in R' to $w(x, y)$.[9]

From (4) one gets for every point $P \equiv (x, y)$ in R'

$$[w(x, y) - w_h(x, y)]^2 \leq \frac{1}{8\pi} \int_c [f - f_h]^2 \, ds \int_c \left(\frac{d\Delta G(P, Q)}{dn}\right)^2 ds$$

$$+ \frac{1}{8\pi} \int_c [g - g_h]^2 \, ds \int_c [\Delta G(P, Q)]^2 \, ds$$

$$\leq \frac{L(R')}{8\pi} \left[\int_c [f - f_h]^2 \, ds + \int_c [g - g_h]^2 \, ds\right],$$

where $L(R')$ is a number larger than the values taken by the following functions of P in R'

$$\int_c \left(\frac{d\Delta G(P, Q)}{dn}\right)^2 d_Q s, \qquad \int_c (\Delta G(P,Q))^2 \, d_Q s.$$

It must be observed that the Green's function has been employed only in order to prove the convergence of $w_h(x, y)$ to $w(x, y)$ but not in the construction of w_h, which requires only the solution of the linear system (5).

2. We will consider now a second procedure, whose theoretical proof is not founded on the use of the Green's function, and which therefore, in addition to the approximate solutions, gives also a proof of the existence theorem.

Our boundary value problem will be formulated in the following way. Let u be a function continuous in $R + C$, together with its first partial derivatives, and such as to make $(\Delta_2 u)^2$ integrable in R.

We will determine a function w satisfying the following conditions:

$$w = u = f$$

$$dw/dn = du/dn = C \text{ on } c, \qquad \Delta\Delta w = 0 \text{ in } R.$$

We will call $\{\psi_k\}$ the set of homogeneous harmonic polynomials;[10] taking

$$\varphi_h = \sum_{k=0}^{h} c_k^{(h)} \psi_k,$$

[9] In the case of a sequence of by-harmonic functions any partial derivative of w_h converges uniformly in the interior of R towards to the corresponding derivatives of $w(x, y)$. See M. Nicolesco: *Les fonctions polyharmoniques*, Hermann, Paris, 1936 and M. Picone: *Sulla convergenza delle successioni di funzioni iperarmoniche*, Bulletin Mathématique de la Société Roumaine des Sciences, t. XXXVIII, 1936.

[10] That is, the system of functions $\psi_0 = 1$, $\psi_{2k-1}(x, y) = \mathcal{R}(x + iy)^k$, $\psi_{2k} = \mathcal{I}(x + iy)^k$ where $k = 1, 2, \cdots$.

the $c_k^{(h)}$, are determined in such a way as to make the value of the following quadratic polynomial in γ_k a minimum

$$\int_R \left[\Delta u - \sum_{k=0}^{h} \gamma_k \psi_k \right]^2 dT^{11},$$

that is, the $c_k^{(h)}$ are solutions of the following system

$$\sum_{j=0}^{h} \gamma_j \int_R \psi_k \psi_j \, dT = \int_R \Delta u \psi_k \, dT \qquad (k = 0, 1, \cdots, h)$$

which, according to Green's formula, can also be written as

$$\sum_{j=0}^{h} \gamma_j \int_R \psi_k \psi_j \, dT = \int_C (g\psi_k - f d\psi_k/dn) \, ds \tag{6}$$

where quantities on the right side are known.

The sequence $\varphi_h(x, y)$ converges in the mean in R and moreover converges uniformly in the interior of R, by a well known property of harmonic functions.

If we call its limit $\varphi(x, y)$, this function will be harmonic in R and its square will therefore be integrable.

The function

$$w(P) = \frac{1}{2\pi} \int_C \left[f(Q) \frac{d \log \overline{POQ}}{dn_Q} - g(Q) \log \overline{PQ} \right] d_Q s + \frac{1}{2\pi} \int \varphi(Q) \log \overline{PQ} \, d_Q T \tag{7}$$

is the required solution of our problem. It satisfies in R the equation $\Delta\Delta w = 0$.

If we call P' any point exterior to R we have

$$0 = \frac{1}{2\pi} \int_C \left[f(Q) \frac{d \log \overline{OP'Q}}{dn_Q} - g(Q) \log \overline{P'Q} \right] d_Q s + \frac{1}{2\pi} \int_R \varphi(Q) \log \overline{P'Q} \, d_Q T. \tag{8}$$

The above relation can be proved as follows.

If we call O the origin of the coordinates and a_k a set of proper constants, the function $\log \overline{PQ}$ admits the following expansion

$$\log \overline{PQ} = a_0 \log \overline{OP} + \sum_{k=1}^{\infty} a_k \frac{\psi_k(Q)\psi_k(P)}{\overline{OP}^{2k}},$$

which for a given P converges uniformly, when Q varies in the interior of the circle of center O and radius $\overline{OP}$; conversely for a given Q the expansion converges uniformly when P varies in the interior of a circle of center O and radius $\overline{OQ}$. It follows that the function $w'(P)$ on the right side of (8) admits the following expansion in the exterior of any circle, having center O and containing R

$$\begin{aligned} w'(P) = &\left[\frac{1}{2\pi} \int_C \left(f \frac{d\psi_0}{dn} - g\psi_0 \right) ds + \frac{1}{2\pi} \int_R \varphi\psi_0 \, dT \right] a_0 \log \overline{OP} \\ &+ \sum_{k=1}^{\infty} \left[\frac{1}{2\pi} \int_C \left(f \frac{d\psi_k}{dn} - g\psi_k \right) ds + \frac{1}{2\pi} \int_R \varphi\psi_k \, dT \right] a_k \frac{\psi_k(P)}{\overline{OP}^{2k}}. \end{aligned} \tag{9}$$

[11] The elementary area $dx\, dy$ is denoted by dT.

Moreover for any $k \geq 0$, we have

$$\int_R \varphi\psi_k \, dT = \lim_{h\to\infty} \int_R \varphi_h \psi_k \, dT = \lim_{h\to\infty} \sum_{j=1}^{h} c_j^{(h)} \int_R \psi_j \psi_k \, dT = \int_C (g\psi_k - f d\psi_k/dn) \, ds.$$

As all the coefficients of the expansion (9) are zero and $w(P')$ is analytic, it follows that (8) is satisfied for every P' exterior to R.

The function $w(P)$ is continuous in $R + C$, under suitable hypotheses on C, by the theory of logarithmic potentials. If we take M on C, we get

$$\lim_{P\to M} w(P) = \tfrac{1}{2} f(M) + \frac{1}{2\pi} \int_C \left[f(Q) \frac{d \log \overline{MQ}}{dn_Q} - g(Q) \log \overline{MQ} \right] d_Q s + \frac{1}{2\pi} \int_R \varphi(Q) \log \overline{MQ} \, d_Q T,$$

while for $P' \to M$ from the exterior of R we get

$$\sigma = -\tfrac{1}{2} f(M) + \frac{1}{2\pi} \int_C \left[f(Q) \frac{d \log \overline{MQ}}{dn_Q} - g(Q) \log \overline{MQ} \right] d_Q s + \frac{1}{2\pi} \int_R \varphi(Q) \log \overline{MQ} \, d_Q T.$$

It follows that

$$\lim_{P\to M} w(P) = f(M).$$

It can be similarly proved that

$$\lim_{P\to M \,(\text{on } n_M)} dw(P)/dn_M = g(M).$$

To this purpose we must first extend to the logarithmic potential of surface distribution on R the theorem on continuity of the normal derivative through C, a theorem, which is known only when $\varphi(Q)$ is bounded in $R + C$. This extension has been obtained in a previous paper[12] and therefore the existence theorem can be proved.

Concerning the evaluation of $w(P)$, it may be pointed out that the sequence

$$w_h(P) = \frac{1}{2\pi} \int_C \left[f(Q) \frac{d \log \overline{PQ}}{dn_Q} - g(Q) \log \overline{PQ} \right] d_Q s + \frac{1}{2\pi} \int_R \varphi_h(Q) \log \overline{PQ} \, d_Q T$$

converges uniformly in R to $w(P)$ and therefore any one of its derivatives of arbitrary order converges towards the corresponding derivative of $w(P)$ in any closed set interior to R. The approximate solution $w_h(P)$ can also be taken as

$$w_h(P) = \frac{1}{8\pi} \int_C \left(f \frac{d\Delta r^2 \log r}{dn} - g \Delta r^2 \log r \right) ds + \frac{1}{8\pi} \int_C \left(\varphi_h \frac{dr^2 \log r}{dn} - \frac{d\varphi_h}{dn} r^2 \log r \right) ds,$$

[12] See G. Fichera, *Teorema d'esistenza per il problema bi-iperarmonico*, Rend. Acc. Lincei, dicembre 1948.

where no integrals of the type $\int_R \varphi_h \log r \, dT$, which are inconvenient from a numerical stand-point, are present.

If we call $\{\Psi_k\}$ the set of $\{\psi_k\}$ orthonormalised—for instance—according to SCHMIDT's procedure, we may take

$$\varphi_h = \sum_{k=0}^{h} c_k \Psi_k ,$$

where

$$c_k = \int_C (g\psi_k - f d\psi_k/dn) \, dT .$$

In general the orthonormal system $\{\Psi_k\}$ is not known and therefore it is usually more convenient to evaluate w by solving the system (6), without any previous orthonormalisation of $\{\psi_k\}$.

The introduction of an orthonormal set is however useful because it permits the determination of a simple and straightforward expression for Green's function for the given problem, which is defined—as is well known—as follows

$$G(P, Q) = \overline{PQ}^2 \log \overline{PQ} + \varpi(P, Q)$$

where $\varpi(P, Q)$ is the function satisfying the following conditions at any point P interior to R.

$$\varpi(P, Q) = -\overline{PQ}^2 \log \overline{PQ}, \qquad \frac{d\varpi(P, Q)}{dn_P} = -\frac{d}{dn_P}(\overline{PQ} \log \overline{PQ}) \qquad \text{on } C,$$

$$\Delta\Delta\varpi(P, Q) = 0 \qquad \text{in } R.$$

Then (7) gives

$$\begin{aligned}
\varpi(P, Q) = {} & \frac{-1}{2\pi} \int_C \left[\overline{QM}^2 \log \overline{QM} \, \frac{d \log \overline{PM}}{dn_M} - \log \overline{PM} \, \frac{d}{dn_M} \overline{QM}^2 \log \overline{QM} \right] d_M T \\
& + \frac{1}{2\pi} \int_R \left\{ \log \overline{PM} \sum_{k=0}^{\infty} \left[\int_C \left(\Psi_k(T) \, \frac{d}{dn_L} \overline{QL}^2 \log \overline{QL} \right. \right. \right. \\
& \left. \left. \left. - \frac{d\Psi_k(L)}{dn_L} \overline{QL}^2 \log \overline{QL} \right) d_L T \right] \Psi_k(M) \right\} d_M T \\
= {} & -\overline{PQ}^2 \log \overline{PQ} + \frac{2}{\pi} \int_R \log \overline{QM} \log \overline{PM} \, d_M T \\
& - \frac{2}{\pi} \sum_{k=0}^{\infty} \int_R \Psi_k(M) \log \overline{PM} \, d_M T \int_R \Psi(M) \log \overline{QM} \, d_M T ,
\end{aligned}$$

and therefore

$$\boxed{\begin{aligned}
G(P, Q) = {} & \frac{2}{\pi} \int_R \log \overline{QM} \log \overline{PM} \, d_M T \\
& - \frac{2}{\pi} \sum_{k=0}^{\infty} \int_R \Psi(M) \log \overline{PM} \, d_M T \int_R \Psi_k(M) \log \overline{QM} \, d_M T
\end{aligned}}$$

From the above considerations the series on the right side converges uniformly for a given Q, when P varies in R and the same holds when P is fixed and Q varies. From the expression obtained for $G(P, Q)$, some properties can be derived of Green's functions for a clamped plate (for instance, the symmetry with respect to P and Q).

3. These properties are usually proved under less general hypotheses for $G(P, Q)$.

There is also another method, by which the existence theorem and computation procedure for the solution can be obtained at the same time. This method has this advantage over the previous method: the approximating function is not given by line-integrals, but by polynomials. The following integral relations are valid

$$\int_C \left(\operatorname{grad} w \times \frac{d}{dn} \operatorname{grad} w' - w \frac{d}{dn} \Delta\, w'\right) d\tau$$

$$- \int_C \left(\operatorname{grad} w' \times \frac{d}{dn} \operatorname{grad} w - w' \frac{d}{dn} \Delta w\right) d\tau$$

$$+ \int_R (w \Delta\Delta w' - w' \Delta\Delta w)\, dT = 0,$$

$$\int_C \left[\operatorname{grad} w \times \frac{d}{dn} \operatorname{grad} w' - w \frac{d}{dn} \Delta\, w'\right] d\tau$$

$$= \int_R (w_{xx} w'_{xx} + 2 w_{xy} w'_{xy} + w_{yy} w'_{yy})\, dT - \int_R w \Delta\Delta w'\, dT.$$

Our boundary value problem requires the determination of a solution of the equation $\Delta\Delta w = 0$, when w and grad w are both given on C. Assume that on C, $w = u$, and grad $w =$ grad u, where the function u is continuous in $R + C$ together with its first derivatives and has second derivatives, whose squares are integrable. Let $\{\omega_k\}$ be a set of by-harmonic homogeneous polynomials of degree ≥ 2 and let us take the origin of coordinates inside R. We must have

$$\int_C \left(\operatorname{grad} \omega_k \times \frac{d}{dn} \operatorname{grad} w - \omega_k \frac{d}{dn} \Delta_2 w\right) dT$$

$$= \int_C \left(\operatorname{grad} u \times \frac{d}{dn} \operatorname{grad} u - u \frac{d}{dn} \Delta_2 \omega_k\right) dT.$$

According to the procedure given by PICONE, let us take $w = \sum_{i=1}^{h} c_i^{(h)} \omega_i$; when this expression is substituted in the last one, we get

$$\sum_{i=1}^{h} c_i^{(h)} \int_C \left(\operatorname{grad} \omega_k \times \frac{d}{dn} \operatorname{grad} \omega_i - \omega_k \frac{d}{dn} \Delta\, \omega_i\right) dT$$

$$= \int_C \left(\operatorname{grad} u \times \frac{d}{dn} \operatorname{grad} \omega_k - u \frac{d}{dn} \Delta\, \omega_k\right) dT \quad (k = 1, 2, \cdots, h)$$

which can be written also as

$$\sum_{i=1}^{h} c_i^{(h)} \int_R (\omega_{xx}^{(h)} \omega_{xx}^{(i)} + 2\omega_{xy}^{(h)} \omega_{xy}^{(i)} + \omega_{yy}^{(h)} \omega_{yy}^{(i)})\, dT$$

$$= \int_R (u_{xx} \omega_{xx}^{(h)} + 2\omega_{xy} \omega_{xy}^{(h)} + u_{yy} \omega_{yy}^{(h)})\, dT. \tag{10}$$

The previous system is obtained, by making the following integral a minimum

$$\int_R \left\{ \left[u_{xx} - \sum_{i=1}^{h} c_i^{(h)} \omega_{xx}^{(i)} \right]^2 + 2 \left[u_{xy} - \sum_{i=1}^{h} c_i^{(h)} \omega_{xy}^{(i)} \right]^2 + \left[u_{yy} - \sum_{i=1}^{h} c_i^{(h)} \omega_{yy}^{(i)} \right]^2 \right\} dT.$$

It follows that, if we take

$$\Omega^{(h)} = \sum_{i=1}^{h} c_i^{(h)} \omega_i,$$

the sequences $\{\Omega_{xx}^{(h)}\}$, $\{\Omega_{xy}^{(h)}\}$, $\{\Omega_{yy}^{(h)}\}$ converge in the mean, and therefore converge uniformly in the interior of R. If we take

$$\varphi_{xx} = \lim_{h \to \infty} \Omega_{xx}^{(h)}, \qquad \varphi_{xy} = \lim_{h \to \infty} \Omega_{xy}^{(h)}, \qquad \varphi_{yy} = \lim_{h \to \infty} \Omega_{yy}^{(h)}$$

the function defined by

$$w(P) = \frac{1}{8\pi} \int_c \left[\operatorname{grad} u \times \frac{d}{dn} \operatorname{grad} (r^2 \log r) - u \frac{d}{dn} \Delta(r^2 \log r) \right] d\tau$$

$$- \frac{1}{8\pi} \int_R \left[\varphi_{xx} \frac{\partial^2 r^2 \log r}{\partial x^2} + 2\varphi_{xy} \frac{\partial^2 r^2 \log r}{\partial x \partial y} + \varphi_{yy} \frac{\partial^2 r^2 \log r}{\partial y^2} \right] dT$$

is the solution of the problem.

This can be proved in the usual way, considering that for P exterior to R, one has

$$0 = \frac{1}{8\pi} \int_c \left[\operatorname{grad} u \times \frac{d}{dn} \operatorname{grad} (r^2 \log r) - u \frac{d}{dn} \Delta_r (r^2 \log r) \right] d\tau$$

$$+ - \frac{1}{8\pi} \int_R \left[\varphi_{xx} \frac{\partial^2 r^2 \log r}{\partial x^2} + 2\varphi_{xy} \frac{\partial^2 r^2 \log r}{\partial x \partial y} + \varphi_{yy} \frac{\partial^2 r^2 \log r}{\partial y^2} \right] dT,$$

as is shown by (10).

Since

$$w_{xx} = \varphi_{xx}, \qquad w_{xy} = \varphi_{xy}, \qquad w_{yy} = \varphi_{yy},$$

it follows also that $\{\Omega^{(h)}\}$ converges towards a function which differs from w by a first degree polynomial.

The methods which have been illustrated can be applied, in addition to the above problem, to a larger class of boundary value problems involving linear partial differential equations.

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MULTIPLE ELLIPTIC INTEGRALS

By E. L. Kaplan

1. Introduction. It is well-known that elliptic integrals occur in many physical problems, and not infrequently in the form of multiple integrals. One of the integrations being performed, there results an integrand which itself involves elliptic integrals.

The class of integrals considered here is

$$(1) \qquad \int Kk^n k'^{n'}\,dk, \qquad \int Ek^n k'^{n'}\,dk,$$

where K and E are the complete elliptic integrals of modulus k, $k'^2 = 1 - k^2$ and n, n' are usually integral. Concerning K and E the reader may consult Whittaker and Watson [1] for theory and Fletcher [2] for references to tables. Since $k\,dk = -k'\,dk'$, (1) differs only in notation from the forms

$$\int \begin{matrix} K \\ E \end{matrix} k^n k'^{n'}\,dk', \qquad \int \begin{matrix} K' \\ E' \end{matrix} k^n k'^{n'}\,\frac{dk}{dk'},$$

where K' and E' have the modulus k'. It will be shown that four particular indefinite integrals suffice for the evaluation of all integrals of these types. The corresponding definite integrals are evaluated, and also ten-decimal values of $\int_0^1 (n+1)k^n K\,dk$ for $n = -0.9(0.1)2.0$.

When n and n' are integers; as will usually be assumed, (1) is reducible to the forms

$$(2) \quad \text{(a)} \int \begin{matrix} K \\ E \end{matrix} k^m\,dk, \quad \text{(b)} \int \begin{matrix} K' \\ E' \end{matrix} k^m\,dk, \quad \text{(c)} \int \begin{matrix} K \\ E \end{matrix} k'^m\,dk, \quad \text{(d)} \int \begin{matrix} K' \\ E' \end{matrix} k'^m\,dk.$$

First (1) is multiplied by a power of $k^2 + k'^2$ ($\equiv 1$) sufficient to make at least one of n, n' non-negative in each of the resulting terms. Suppose that $n \geq 0$. If n is even, $k^n = (1 - k'^2)^{n/2}$ may be expanded in powers of k', yielding type (c) integrals. If n is odd, replace $k\,dk$ by $-k'\,dk'$, expand k^{n-1} in powers of k', and write k, K', E' for k', K, E respectively. If $n' \geq 0$ the procedure is similar, and integrals of type (a) or (d) are obtained. In formula designations the letters (a) to (d) will always parallel their use in (2).

The paper of Spielrein [3] may be mentioned as an example of the occurrence of these integrals.

2. Reduction Formulas. These were obtained by eliminating unwanted terms from among relations obtained by differentiation, using the results

$$(3) \qquad \frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}, \qquad \frac{dE}{dk} = \frac{E - K}{k}.$$

(4a) $$k^{n-1}(nk'^2K - E) = (n-1)^2 \int k^{n-2}K\,dk - n^2 \int k^n K\,dk.$$

$$k^{n-1}[k'^2K + (n-2)E - nk^2E] = (n-1)^2 \int k^{n-2}E\,dk - n(n+2) \int k^n E\,dk.$$

$$k^{n+1}E = (n+2) \int k^n E\,dk - \int k^n K\,dk.$$

(4b) $$k^{n-1}k'^2[E' + (n-1)K' - (n+2)k^2K'] = (n-1)^2 \int k^{n-2}K'\,dk$$

$$- (2n^2 + 2n + 1) \int k^n K'\,dk + (n+2)^2 \int k^{n+2}K'\,dk.$$

$$k^{n-1}k'^2[k^2K' - (n+1)E' + (n+2)k^2E'] = -(n^2-1) \int k^{n-2}E'\,dk$$

$$+ (2n^2 + 6n + 3) \int k^n E'\,dk - (n+2)(n+4) \int k^{n+2}E'\,dk.$$

(4c) $$kk'^n[E + nK - (n+2)k'^2K] = -n^2 \int Kk'^{n-2}\,dk$$

$$+ (2n^2 + 4n + 3) \int Kk'^n\,dk - (n+2)^2 \int Kk'^{n+2}\,dk.$$

$$kk'^n[k'^2K - (n+2)k^2E] = n(n+2) \int Ek'^{n-2}\,dk$$

$$- (2n^2 + 8n + 7) \int Ek'^n\,dk + (n+2)(n+4) \int Ek'^{n+2}\,dk.$$

(4d) $$kk'^n[E' - (n+1)K' + (n+2)k'^2K'] = n^2 \int K'k'^{n-2}\,dk$$

$$- (2n^2 + 4n + 3) \int K'k'^n\,dk + (n+2)^2 \int K'k'^{n+2}\,dk.$$

$$kk'^n[k^2K' + (n-1)E' - (n+2)E'k'^2] = -n^2 \int E'k'^{n-2}\,dk$$

$$+ (2n^2 + 6n + 5) \int E'k'^n\,dk - (n+2)(n+4) \int E'k'^{n+2}\,dk.$$

Many integration formulas are derivable as special cases of the foregoing, and will therefore not be repeated. In addition there are the following:

(5b) $$\int kK'\,dk = -E' + k^2K', \qquad \int E'\,dk/k = E' - K',$$

$$\int K'\,dk/k^2 = (E' - K')/k, \qquad \int E'\,dk/k^2 = kK' - E'/k,$$

$$2\int E'\,dk = \int K'\,dk + k(E' - K').$$

(5c)
$$\int k'K\,dk = \int K\,dk/k' - \int E\,dk/k' + kk'K,$$

$$3\int k'E\,dk = \int K\,dk/k' + kk'(K + E),$$

$$\int E\,dk/k'^2 = kK, \qquad 2\int E\,dk/k'^4 = kE/k'^2 + \int K\,dk/k'^2.$$

(5d)
$$\int_0^\kappa k'K'\,dk = \int_{\kappa'}^1 E\,dk/k' - kk'K\Big|_{k=\kappa'}^{k=1}$$

$$3\int_0^\kappa k'E'\,dk = 3\int_{\kappa'}^1 E\,dk/k' - \int_{\kappa'}^1 K\,dk/k' - kk'(K + E)\Big|_{k=\kappa'}^{k=1}$$

$$\int_0^\kappa K'\,dk/k' = \int_{\kappa'}^1 K\,dk/k', \qquad \int_0^\kappa E'\,dk/k' = \int_{\kappa'}^1 E\,dk/k',$$

$$\int E'\,dk/k'^2 = \int K'\,dk/k'^2 - kK'.$$

A few of these formulas occur in Jahnke-Emde [4].

3. Use of Landen's Transformation. The transformation formula is

(6)
$$K(k) = \frac{1}{1+k}\cdot K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2}{1+k'}\cdot K\left(\frac{1-k'}{1+k'}\right),$$

which yields at least six new integration formulas. There is a similar (more complicated) formula for E, but it seems not to yield anything additional. If $\lambda = 2\sqrt{\kappa}/(1+\kappa)$, the results are

(7a)
$$\int_\kappa^1 K\,dk = \left[\int_\kappa^1 - 2\int_\lambda^1\right] K\,dk/k,$$

(7b)
$$\int_\kappa^1 K'\,dk = \left[\int_\kappa^1 - 4\int_\lambda^1\right] K'\,dk/k,$$

(7c)
$$\int_0^\kappa K\,dk/k'^2 = \left[\tfrac{1}{2}\int_{\lambda'}^1 - \int_{\kappa'}^1\right] K'\,dk/k,$$

(7d)
$$\int_0^\kappa K'\,dk/k'^2 = \left[\int_{\lambda'}^1 - \int_{\kappa'}^1\right] K\,dk/k.$$

(8)
$$\int_0^\kappa K\,dk/\sqrt{k} = \int_0^\lambda K\,dk/k',$$

(9)
$$\int_0^\kappa \sqrt{k}K\,dk = \int_0^\lambda (K - 2E)\,dk/k' + 2\sqrt{\kappa}E(\lambda).$$

In (7a, b) one begins by transforming the $\int_\lambda^1$. In the others one transforms the left member, making use of (7b), (7a), and (4d) in getting (7c), (7d), and (9) respectively.

4. The Standard Integrals. The preceding work suffices to show that four particular integrals will serve for the evaluation of the entire class of integrals here considered. These four may be taken as

$$\int K\,dk/k, \qquad \int K'\,dk/k, \qquad \int K\,dk/k', \qquad \int E\,dk/k'. \tag{10}$$

If one does not wish to employ Landen's transformation, then it appears necessary to add four more integrals to the list, namely

$$\int K\,dk, \qquad \int K'\,dk, \qquad \int K\,dk/k'^2, \qquad \int K'\,dk/k'^2. \tag{11}$$

In addition, Landen's transformation brings the forms $\int \frac{K}{E}\,k^{n+1/2}\,dk$ within the fold.

5. Definite Integrals. All of the integrals considered (modified where necessary to produce convergence) with the limits 0 and 1 can be expressed in terms of known constants. These results, some of which are well-known, and included only for completeness, are not only interesting in themselves but would be a valuable aid in the construction of tables of the indefinite integrals.

$$\int_0^1 K\,dk = 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) = 2G, \qquad \int_0^1 E\,dk = G + \tfrac{1}{2}, \tag{12}$$

$$\int_0^1 K'\,dk = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \pi^2/4, \qquad \int_0^1 E'\,dk = \pi^2/8, \tag{13}$$

$$\begin{aligned} \int_0^1 (K - \tfrac{1}{2}\pi)\,dk/k &= \pi \log 2 - 2G = .34565\,49019, \\ \int_0^1 (E - \tfrac{1}{2}\pi)\,dk/k &= \pi \log 2 - 2G + 1 - \tfrac{1}{2}\pi, \end{aligned} \tag{14}$$

$$\begin{aligned} \int_0^1 (K' - \log 4/k)\,dk/k &= 2\log^2 2 - \pi^2/12 = .13843\,89944, \\ \int_0^1 (E' - 1)\,dk/k &= 2\log 2 - 1, \end{aligned} \tag{15}$$

$$\begin{aligned} \int_0^1 K\,dk/k' &= \int_0^{\pi/2}\int_0^{\pi/2} (1 - \sin^2\theta \sin^2\varphi)^{-1/2}\,d\theta\,d\varphi \\ &= 3.43759\,29090\cdots = K^* = \Gamma^4(\tfrac{1}{4})/16\pi = K^2(1/\sqrt{2}), \end{aligned} \tag{16}$$

$$(17)\quad \int_0^1 E\,dk/k' = \int_0^{\pi/2}\int_0^{\pi/2} (1-\sin^2\theta\sin^2\varphi)^{1/2}\,d\theta\,d\varphi$$
$$= 2.07768\ 14600\cdots = E^* = (K^{*2}+\pi^2/4)/2K^*.$$

G is known as Catalan's constant and has the value 0.91596 55942 to ten decimals, while $\Gamma(1/4) = 3.62560\ 99082$ and $\pi^2/8 = 1.23370\ 05501$.

(12) and (13) are obtained by writing the left member as a double integral and inverting the order of integration, giving

$$\int_0^1 K\,dk = \int_0^{\pi/2}\frac{\varphi\,d\varphi}{\sin\varphi} = 2\int_0^1 \frac{\tan^{-1}t}{t}\,dt = 2\int_0^1\left(1-\frac{t^2}{3}+\frac{t^4}{5}-\cdots\right)dt.$$

$$\int_0^1 K'\,dk = \int_0^{\pi/2}\log\sqrt{\frac{1+\sin\varphi}{1-\sin\varphi}}\cdot\frac{d\varphi}{\sin\varphi}$$
$$= \int_0^{\pi/2}(1+\tfrac{1}{3}\sin^2\varphi+\tfrac{1}{5}\sin^4\varphi+\cdots)\,d\varphi$$
$$= \frac{\pi}{2}\left(1+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{5}\cdot\frac{1.3}{2.4}+\cdots\right) = \frac{\pi}{2}\sin^{-1}1 = \frac{\pi^2}{4}.$$

This integral may also be written as

$$\int_0^{\pi/2}\log\operatorname{ctn}\frac{\varphi}{2}\cdot\frac{d\varphi}{\cos\varphi} = 2\int_0^1\log\frac{1}{t}\cdot\frac{dt}{1-t^2} = \int_0^1\log\frac{1+t}{1-t}\cdot\frac{dt}{t}$$
$$= 2\int_0^1\left(1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots\right)dt = 2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right).$$

(14) is derived from (12) by letting $\kappa\to 0$ in (7a). (15) is similarly derived from (13) and (7b). The second formula of each pair is obtained from the relations among the indefinite integrals.

(16) and (17) are derived by using (8) and (9) and other indefinite integral formulas. Thus we have

$$K^* = \int_0^1 K\,dk/\sqrt{k} = \int_0^1\int_0^1 dk\,dx/\sqrt{k}\sqrt{1-x^2}\sqrt{1-k^2x^2}$$
$$= \int_0^1\int_0^x dy\,dx/\sqrt{xy}\sqrt{1-x^2}\sqrt{1-y^2} = \tfrac{1}{2}\left[\int_0^1 dx/\sqrt{x(1-x^2)}\right]^2$$
$$= \left[\int_0^1 dt/\sqrt{1-t^2}\sqrt{1-t^2/2}\right]^2 = K^2(1/\sqrt{2})\text{ by putting }x = 1-t^2.$$

On the other hand, putting $x=\sqrt{u}$ gives a complete beta function expressible in terms of $\Gamma(\frac{1}{4})$. For (17),

$$[2E(1/\sqrt{2})-K(1/\sqrt{2})]^2 = \left[\int_0^1\sqrt{1-t^2}\,dt/\sqrt{1-t^2/2}\right]^2$$
$$= \tfrac{1}{2}\left[\int_0^1\sqrt{x}\,dx/\sqrt{1-x^2}\right]^2 = \int_0^1\int_0^x\sqrt{xy}\,dy\,dx/\sqrt{1-x^2}\sqrt{1-y^2}$$

$$= \int_0^1\int_0^1 x^2\sqrt{k}\,dk\,dx/\sqrt{1-x^2}\sqrt{1-k^2x^2} = \int_0^1 (K-E)\,dk/k^{3/2}$$

$$= 2(k^2E + E - k'^2K)k^{-1/2}\Big|_{k=0}^{k=1} - \int_0^1 \sqrt{k}(5E-K)\,dk \quad \text{by (4a)}$$

$$= 4 - 2k^{3/2}E\Big|_0^1 - \int_0^1 \sqrt{k}K\,dk = 2E^* - K^* \quad \text{by (9)}.$$

Legendre's relation $EK' + E'K - KK' = \pi/2$ gives

$$K(1/\sqrt{2})[2E(1/\sqrt{2}) - K(1/\sqrt{2})] = \pi/2,$$

since primed and unprimed quantities are here the same. Hence $K^*(2E^* - K^*) = (\pi/2)^2$, and (17) follows.

Various integrals of the form

$$(18) \qquad f(a, b; c) = \int_0^1\int_0^1 (1-x^2)^{a/2}(1-y^2)^{b/2}(1-x^2y^2)^{c/2}\,dx\,dy$$

are expressible in terms of K^* and E^*, thus:

$$f(-1,-1;-1) = K^*, \quad f(-1,1;-1) = K^* - E^*, \quad f(1,1;-1) = 2K^* - 3E^*,$$

$$f(-1,-1;1) = E^*, \quad f(-1,1;1) = K^*/3, \quad f(1,1;1) = 7K^*/9 - E^*.$$

An integral related to the first of these, together with two other more complicated integrals, has been evaluated by G. N. Watson [5].

Before it was found how to express K^* and E^* in terms of known constants, they were evaluated by what may be called asymptotic interpolation. The function

$$(19) \qquad M(n) = (n+1)\int_0^1 k^n K\,dk \qquad (\text{for } n > -1)$$

has the recurrence formula

$$(20) \qquad n^2M(n) = (n^2-1)M(n-2) + n + 1$$

and the particular values $M(-3) = \pm\infty$, $M(-2) = 1$, $M(-1) = \pi/2$, $M(0) = 2G$, $M(1) = 2$. It was calculated as far as $n = 31$, which permitted interpolation for $M(23\frac{1}{2})$ and $M(24\frac{1}{2})$ to about 12 decimals. Reverse applications of (20) gave $M(\pm 1/2)$, and finally

$$(21) \qquad K^* = 2M(-1/2),\ E^* = M(-1/2) - M(1/2)/3 + 1.$$

Much of the necessary work having already been done, additional values of

$M(n)$ (see (19)) were interpolated as follows:

n	$M(n)$	n	$M(n)$
−0.9	1.60420 29030	0.6	1.93969 61613
−0.8	1.63547 89031	0.7	1.95550 39672
−0.7	1.66487 03305	0.8	1.97080 32532
−0.6	1.69258 42642	0.9	1.98562 56612
−0.5	1.71879 64545	1.0	2.00000 00000
−0.4	1.74365 71874	1.1	2.01395 25699
−0.3	1.76729 58642	1.2	2.02750 74426
−0.2	1.78982 46138	1.3	2.04068 67030
−0.1	1.81134 11708	1.4	2.05351 06600
0	1.83193 11884	1.5	2.06599 80303
0.1	1.85167 01114	1.6	2.07816 60986
0.2	1.87062 47073	1.7	2.09003 08593
0.3	1.88885 43234	1.8	2.10160 71404
0.4	1.90641 19292	1.9	2.11290 87135
0.5	1.92334 49834	2.0	2.12394 83913

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PLANE ROTATIONAL PRANDTL-MEYER FLOWS*

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Introduction. We consider the steady, plane flow of a gas subjected to no external forces under the assumption that any effects due to heat conduction radiation, or frictional forces may be neglected.

Among the irrotational flows of a polytropic gas the Prandtl-Meyer flows[1] hold a position of great importance in Gas Dynamics. It seems worth while, therefore, to inquire whether this type of flow permits a generalization to rotational flows. The classic Prandtl-Meyer flows are characterized by a constant velocity vector along each line of a one parameter family of straight lines in the physical plane. Thus, if u, v denote the components of the velocity vector and τ is the parameter in the family of lines, we may write

$$h: \qquad u = u(\tau), \qquad v = v(\tau).$$

Turning to the hodograph plane, the hodograph of the flow in the physical plane appears as a single curve h in the hodograph plane, and we are led to define a *Prandtl-Meyer flow* to be a flow in the physical plane whose hodograph degenerates into a single curve h in the hodograph plane. We obtain in this way a direct generalization of classical Prandtl-Meyer flows, in which the flow is irrotational, to rotational flows.

The following Prandtl-Meyer flows will be excluded as trivial:

HODOGRAPH PLANE		STREAMLINES IN PHYSICAL PLANE
h a segment of a radial line	$\theta = const.$,	*parallel straight lines,*
h an arc of a circle	$q = const.$,	*concentric circles.*[2]

The case in which h is a point and the flow in the physical plane proceeds with constant speed along parallel lines is included above. We accordingly confine our attention to Prandtl-Meyer flows for which the equation of the hodograph curve h can be written

$$h: \qquad \theta = \theta(q) \not\equiv \text{const.}, \qquad q \not\equiv \text{const.},$$

where q, θ denote polar coordinates in the hodograph plane.

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[1] Meyer, T., *Über zweidimensionale Bewegungsvorgänge in einem Gas, das mit Überschallgeschwindigkeit strömt*, Dissertation, Göttingen (1908) or Forschungsheft 62 des Vereins Deutscher Ingenieure, Berlin (1908) pp. 31–67.

Courant, R., Friedrichs, K. O., *Supersonic Flow and Shock Waves*, New York (1948) pp. 262–293.

[2] For a perfect gas this follows from a theorem due to Nemenyi and Prim. See *Some Properties of Rotational Flow of a perfect Gas*, by these authors in the Proceedings of the

Instead of restricting ourselves exclusively to polytropic gases we consider a class of fluids termed *physical gases*, defined in §3, which includes the polytropic gases, reserving a detailed study of rotational Prandtl-Meyer flows of a polytropic gas for a later paper.

1. Confluent flows. Before taking up the study of Prandtl-Meyer flows, we shall make a few remarks which serve to orient these flows among the totality of possible flows.

Consider the following five families of curves in the physical plane:

1. paths of fluid particles = stream lines (ψ = const.),
2. curves of constant pressure = isobars (p = const.),
3. curves of constant speed = isovels (q = const.),
4. curves of constant direction = isoclines (θ = const.),
5. curves of constant density = isopycnics (ρ = const.).

Flows for which any two of these five families coincide are termed *confluent*, otherwise *fluent*. Prandtl-Meyer flows, or briefly *PM-flows* are confluent flows, since the isoclines and isovels coincide in view of $\theta = \theta(q)$. In addition, we can, *a priori*, distinguish nine more types of confluent flows as listed in the table below.

Confluent Flows

	STREAM LINES 1	ISOBARS 2	ISOVELS 3	ISOCLINES 4	ISOPYCNICS 5
1 Stream Lines	* * *	$p = p(\psi)$ concentric circles or straight lines[3]	$q = q(\psi)$ concentric circles or parallel straight lines[4]	$\theta = \theta(\psi)$ Rectilinear[5]	$p = p(\psi)$ Incompressible
2 Isobars	$p = p(\psi)$ concentric circles or parallel straight lines[3]	* * *	$q = q(p)$ Irrotational	$\theta = \theta(p)$ Isoclinic[6]	$\rho = \rho(p)$ Isentropic
3 Isovels	$q = q(\psi)$ concentric circles or parallel straight lines[4]	$q = q(p)$ Irrotational	* * *	$\theta = \theta(q)$ PM-flows	$\rho = \rho(q)$

National Academy of Sciences, vol. 34. (1948) pp. 119–124, in particular Theorem 1. For a general fluid, in which it is assumed that the density is a function of the pressure along each stream line, see a paper. *A new Approach to Problems in two-dimensional Gas Flow* by the author, to appear in the Quarterly of Applied Mathematics.

[3] See §3 of the paper by the author cited in 2.

[4] For a perfect gas see Nemenyi, P., Prim, R. C. loc cit.[2]

Confluent Flows (Cont.)

	STREAM LINES 1	ISOBARS 2	ISOVELS 3	ISOCLINES 4	ISOPYCNICS 5
4 Isoclines	$\theta = \theta(\psi)$ Rectilinear[5]	$\theta = \theta(p)$ Isoclinic[6]	$\theta = \theta(q)$ PM-flows	* * *	$\theta = \theta(\rho)$
5 Isopycnics	$\rho = \rho(\psi)$ Incompressible	$\rho = \rho(p)$ Isentropic	$\rho = \rho(q)$	$\theta = \theta(\rho)$	* * *

A numerical notation for confluent flows suggests itself. Thus we can designate a PM-flow by [34] or [43]. The ten types of confluent flows are not necessarily mutually exclusive. Indeed, if we assume $pq \neq 0$, it follows from

$$pq(q_u u + q_v v) = -(p_u u + p_v v),$$

that [12] and [13] imply each other. If [12] is excluded, irrotational flows are characterized[7] by $q = q(p)$, i.e., by the coincidence of the isobars and isovels, and these flows may be designated by [23]. Again excluding [12], it can be shown that in general [23] implies [25], but conversely, [25] alone does not imply [23]. Various subclasses and simple relations are possible, since more than two families can coincide, e.g., [25] [34] represents *isentropic, PM-flows* and [23] [34] *irrotational, PM-flows*. In general, any two of [23], [24], [34] imply the third.

We shall, however, not pursue the classification and study of the relations between confluent flows and their subclasses any further, since it is the purpose of this paper to study a single type of confluent flow, namely PM-flows.

2. General flows. If, (a) the isobars and stream lines do not coincide, i.e., [12] is excluded; (b) the isobars and isoclines do not coincide, i.e., [24] is excluded[8]; and (c) the speed q, the density ρ, the pressure gradient dp/ds along a stream line remain finite without vanishing, the flow is said to be *general*. The distinction between general and fluent flows should be emphasized. In a fluent flow all ten types of confluent flows are excluded; in a general flow we require that ρ, q, dp/ds remain finite but never vanish, and the confluent flows [12], [24] only are excluded. In particular [34] is not excluded and general flows of type [34], are termed *general PM-flows* or *GPM-flows*.

The problem of determining the general flow of a fluid can be reduced[9] to

[5] Martin, M. H. *Steady, rotational, plane flow of a gas*, Naval Ordnance Laboratory Memorandum 10274, in particular §6.

[6] Loc. cit. 5, in particular §5.

[7] Loc. cit., 5, in particular Theorem 2.

[8] When this procedure is compared with the hodograph method widely used in the study of irrotational flows, which amounts to assuming that the isovels and isoclines do not coincide so that they may be used as curvilinear coordinate lines, it raises the possibility that the use of other pairs from the five families in §1 as curvilinear coordinates may prove of value in the study of confluent and fluent flows.

[9] Loc. cit. 5, in particular §2 and §4.

finding two functions

$$q(p, \psi) \equiv \text{the } \textit{Bernoulli function}; \qquad \theta(p, \psi) \equiv \text{the } \textit{direction function}$$

which satisfy the equation

$$q\left(\frac{q_{pp} - q\theta_p^2}{\theta_\psi}\right)_\psi + (q^2\theta_p)_p = 0. \tag{1}$$

This equation plays a double rôle in the theory, offering a direct, or an inverse approach, to the problem of determining general flows according as, the Bernoulli function q, or the direction function θ, is taken to be a preassigned function of p, ψ.

In the direct approach, (1) becomes a quasi-linear partial differential equation for the determination of the unknown direction function $\theta(p, \psi)$. Thus if the equation of state $\rho = f(p, S)$ for the gas and the entropy distribution function $S = S(\psi)$ are given, $\rho = \rho(p, \psi)$ is a known function termed the *density function*, According to Bernoulli's theorem, i.e.,

$$\tfrac{1}{2}q^2 = C - \int_{p_1}^{p} \rho^{-1}\, dp, \qquad p_1 = p_1(\psi), \qquad C = C(\psi), \tag{2}$$

the function $q = q(p, \psi)$ may be specified by fixing the arbitrary functions $p_1(\psi)$, $C(\psi)$. Whenever the equation of state permits it, we may let $p_1 \to 0$ in (2) and determine the distribution of energy from stream line to stream line by the choice of the arbitrary function $C(\psi)$, which, for this reason, is termed the *energy distribution function*[10].

Alternatively, if $q(p, \psi)$ is prescribed, the density function is

$$\rho = -\,(qq_p)^{-1} = \rho(p, \psi), \tag{3}$$

from which an equation of state is obtained by setting $\psi = \psi(S)$, an arbitrary function of S. The entropy distribution function $S(\psi)$ is defined implicitly by this equation, and the energy distribution function by

$$C(\psi) = \lim_{p \to 0} \tfrac{1}{2}q^2,$$

provided, of course, this limit exists.

Under the inverse approach, (1) becomes a linear partial differential equation of the third order for the unknown Bernoulli function $q(p, \psi)$. Given a solution $q(p, \psi)$ of this partial differential equation, the density function, equation of state, the entropy and energy distribution functions are determined as above.

With either approach, once we are in possession of a pair of functions $q(p, \psi)$, $\theta(p, \psi)$ satisfying (1), a general flow is presented[11] in the physical plane by

[10] C is, of course, Bernoulli's constant and is frequently termed the "stagnation enthalpy." We adopt the above terminology to emphasize the change of C from one stream line to another in the "weak form" of Bernoulli's Theorem.

[11] See §4, loc. cit. 5.

$$
\begin{aligned}
x &= \int \left\{ -\frac{q_{pp} - q\theta_p^2}{\theta_\psi} \cos\theta \, dp + (q \sin\theta)_p \, d\psi \right\}, \\
y &= \int \left\{ -\frac{q_{pp} - q\theta_p^2}{\theta_\psi} \sin\theta \, dp - (q \cos\theta)_p \, d\psi \right\},
\end{aligned} \tag{4}
$$

and in the hodograph plane by

$$
u = q \cos\theta, \qquad v = q \sin\theta. \tag{5}
$$

Depending on whether we keep ψ fixed and let p vary, or keep p fixed and let ψ vary, these equations present the stream lines, or isobars, parametrically in the two planes. To obtain the isovels, isoclines, and isopycnics in the two planes, one assigns constant values to q, θ, qq_p respectively in (4), (5).

3. The Bernoulli function. From the mathematical standpoint we may take the function f in the equation of state $\rho = f(p, S)$ to be an arbitrary function of its arguments, subject, of course, to any assumptions of continuity that may be necessary to the mathematical treatment[12]. Further restrictions on f are, however, dictated by physical considerations. In addition to taking f to be continuous we impose the following conditions upon it:

$$
\begin{aligned}
&\text{(i)} \quad f(0, S) \equiv 0; \qquad 0 < f(p, S) < +\infty, && \text{for } p > 0, \\
&\text{(ii)} \quad f_S(p, S) < 0, \qquad f_p(p, S) > 0, && \text{for } p > 0, \\
&\text{(iii)} \quad i = \int_0^p f^{-1}(p, S)\, dp < +\infty, && \text{for } p > 0.
\end{aligned} \tag{6}
$$

(i) implies a finite, positive density which vanishes only if $p = 0$, and if $p = 0$, is excluded, (ii) implies that, for fixed pressure, the density decreases with increasing entropy and increases, for constant entropy, as the pressure increases. This implies that the pressure increases with the density when the entropy is constant and also increases with the entropy when the density is held constant[13]. (iii) implies a finite enthalpy i, or heat content per unit mass under adiabatic processes, and permits one to let $p_1 \to 0$ in (2.)

If the function f in the equation of state meets the conditions (6), we shall call the gas a *physical gas*. If these assumptions are not made on f and q is taken to be any arbitrarily prescribed function of p, ψ, we shall speak of an *abstract fluid*. Since physical gases form a subclass of abstract fluids, any result obtained for the flow of an abstract fluid automatically applies to the flow of a physical gas. The development of a theory for abstract fluid flows[14] is of more than academic interest, since it would signalize universal properties of fluid flows not dependent on the specializing assumptions (6).

This paper will be restricted, however, to the study of physical gases and in this section we shall investigate the influence of assumptions (6) on the Bernoulli function q.

[12] This is the viewpoint adopted in the paper cited under 5.

[13] Cf. Courant, R., Friedrichs, K. O. op. cit., p. 5.

[14] For a beginning in this direction see 5.

We begin by deriving necessary conditions on q implied by (6). In view of (6 iii) we can let $p_1 \to 0$ in (2), and if we assign the entropy and energy distribution functions $S(\psi)$, $C(\psi)$, the Bernoulli function q is defined by

$$\tfrac{1}{2}q^2 = C(\psi) - \int_0^p f^{-1}(p, S(\psi))\, dp.$$

If $C(\psi) = \text{const.}$, the flow is *isoenergetic*, otherwise *non-isoenergetic*. If $S(\psi) = \text{const.}$, the flow is *isentropic*[15]; if $S'(\psi) \not\equiv 0$, the flow is *anisentropic*.

Taking partial derivatives, we find

$$qq_p = -f^{-1}, \qquad (qq_p)_p = f^{-2}f_p, \qquad (qq_p)_\psi = f^{-2}f_S S',$$

where here, and throughout the paper, the derivative of a function of a single variable is indicated by affixing a prime. In view of (6) we see that for a general flow p is restricted to the range $0 < p < p_0$ where $p_0 = p_0(\psi)$ is the *stagnation pressure* defined by

$$\int_0^{p_0} f^{-1}(p, S(\psi))\, dp = C(\psi),$$

and that $q \geqq 0$ must meet the following conditions:

$$\text{(7)} \quad \begin{aligned} &\text{(i)} && \lim_{p\to 0} qq_p = -\infty; && -\infty < qq_p < 0, && \text{for } 0 < p < p_0, \\ &\text{(ii)} && (qq_p)_\psi \not\equiv 0; && (qq_p)_p > 0, && \text{for } 0 < p < p_0, \\ &\text{(iii)} && \lim_{p\to 0} \tfrac{1}{2}q^2 = C(\psi) && \tfrac{1}{2}q^2 < C(\psi), && \text{for } 0 < p < p_0. \end{aligned}$$

provided the flow is anisentropic. For isentropic flows the first condition in (ii) is replaced by $(qq_p)_\psi \equiv 0$.

These conditions are also sufficient, for if q satisfies the conditions (7), we use (7i) and (3) to define the density function

$$\rho(0, \psi) \equiv 0, \qquad \rho(p, \psi) = -(qq_p)^{-1}, \qquad \text{for } p > 0.$$

To obtain an equation of state, we set $\psi = \psi(S)$. Clearly (7i) implies (6i) and (7ii) implies (6ii), provided we insist that $\psi'(S)$ and $(qq_p)_\psi$ have opposite signs for anisentropic flows. If $(qq_p)_\psi \equiv 0$, one has $\rho = \rho(p)$, and the flow is isentropic. Finally (6iii) follows from (7iii) if we let $p_1 \to 0$ in

$$\int_{p_1}^{p} qq_p\, dp = \tfrac{1}{2}q^2(p, \psi) - \tfrac{1}{2}q^2(p_1, \psi).$$

The following theorem has been proved.

THEOREM 1. *The necessary and sufficient condition for $q(p, \psi)$ to be the Bernoulli function for anisentropic flow of a physical gas with a prescribed energy distribution function $C(\psi)$, is that $q(p, \psi)$ satisfy the conditions* (7). *For isentropic flows the condition $(qq_p)_\psi \equiv 0$ replaces the condition $(qq_p)_\psi \not\equiv 0$ in* (7*ii*).

[15] That is, of the confluent type designated by [25] in §1.

The general flows of a physical gas can be classified into four broad categories:

I. *Isoenergetic, isentropic flows*[16]. (*Irrotational*),

II. *Isoenergetic, anisentropic flows*. (*Rotational*),

III. *Non-isoenergetic, isentropic flows*. (*Rotational*).

IV. *Non-isoenergetic, anisentropic flows*. (*Rotational*).

In the next section we take up the general theory of GPM-flows and in the following sections we consider GPM-flows of categories II and III. GPM flows of category IV lie outside the scope of this investigation.

4. GPM-flows. Direct approach. A GPM-flow is necessarily rotational and its hodograph curve h must have a polar equation of the form

$$h: \qquad \theta = \theta(q) \not\equiv \text{const.}, \qquad q \not\equiv \text{const.},$$

proposed in the introduction. To see this, we note that flows for which $\theta = \text{const.}$ are confluent flows of type [24]; while flows for which $q = \text{const.}$ are confluent flows of type [13], and therefore, as has been pointed out in §1, come under type [12]. Since confluent flows of type [12] or [24] are excluded in the definition of general flows, the polar equation of h must have the stated form. Finally the flow must be rotational, for, as has been noted in §1, an irrotational GPM-flow[16] is an isoclinic flow, i.e., [23] and [34] imply [24], and a general flow cannot be a confluent flow of type [24].

The Bernoulli function $q(p, \psi)$ is assumed to be given subject to the conditions $q_\psi \not\equiv 0$, $q_{pp} \not\equiv 0$. The first condition insures that a general flow is rotational and the second condition rules out flows for which the Bernoulli function is a linear function of p, say $q = Ap + B$, where A, B denote functions of ψ. These flows hold a very special position in the theory and, from (3), imply an equation of state of the form $\rho = \sigma_1(\sigma_2 - p)^{-1}$, where σ_1, σ_2 denote functions of S. Such fluids do not fall under physical gases because of (6i). In this section no further assumptions, other than obvious continuity assumptions, will be made on q. Consequently the results, in addition to applying to rotational flows of physical gases, apply to rotational flows of abstract fluids.

For a GPM-flow it is necessary and sufficient that (1) permit solutions of the form $\theta = \theta(q)$. Replacing θ by $\theta(q)$ in (1) and writing $\omega = \theta'$, the necessary and sufficient condition on $\theta(q)$ is that the equation

$$\omega' - a\omega - b\omega^3 = 0, \qquad a = a(p, \psi), \qquad b = b(p, \psi), \qquad \omega = \omega(q), \tag{8}$$

where

$$a = (q_{pp}/q_\psi)_\psi \cdot (1/q_{pp})$$
$$b = [(q_\psi^3 q_{pp} - 2q_p q_\psi q_{p\psi} + q_p^2 q_{\psi\psi})q + q_p^2 q_\psi^2]q_\psi^{-2} q_{pp}^{-1}, \tag{9}$$

holds identically in p, ψ, after setting $q = q(p, \psi)$ in ω and ω'.

Since $q = q(p, \psi)$ and $q_\psi \not\equiv 0$, we can eliminate ψ from a, b to replace (8) by

$$\omega' - \alpha\omega - \beta\omega^3 = 0, \qquad \alpha = \alpha(p, q), \qquad \beta = \beta(p, q), \qquad \omega = \omega(q), \tag{10}$$

[16] See, for example, the works cited in 1, 2, 5.

where α, β are prescribed functions and $\omega(q)$ is to be determined so that (10) is an identity in p, q.

Since $\theta \not\equiv$ const., we exclude $\omega \equiv 0$, differentiate (10) partially with respect to p, an infer[17] that $\alpha_p + \beta_p\omega^2 \equiv 0$. If $\beta_p \equiv 0$, necessarily[17] $\alpha_p \equiv 0$, and conversely, so that α, β are functions of q alone, and (10) may be regarded as an ordinary differential equation of the Bernoulli type, the solution of which constitutes exactly the totality of function $\omega(q)$ sought. If $\beta_p \not\equiv 0$, for (10) to be an identity in p, q, at least one function $\omega_0(q) \not\equiv 0$ must exist such that $\omega_0' - \alpha\omega_0 - \beta\omega_0^3 \equiv 0$, i.e., the functions α, β cannot be taken arbitrarily. If $\omega(q) \not\equiv 0$ is a second such function, i.e., if $\omega' - \alpha\omega - \beta\omega^3 \equiv 0$, we eliminate α from the two equations and obtain

$$\omega\omega_0' - \omega_0\omega' - \beta\omega\omega_0(\omega_0^2 - \omega^2) \equiv 0,$$

from which $\beta_p\omega\omega_0(\omega_0^2 - \omega^2) \equiv 0$ results by partial differentiation with respect to p, to imply $\omega = \pm\omega_0$.

These results are summed up in the lemma.

LEMMA 1. *The case $\omega \equiv 0$ being excluded, the necessary and sufficient condition for* (10) *to hold identically in p, q is that one of the following alternatives holds:*

A. *Both functions α, β depend only on q and ω is any solution of the ordinary differential equation of Bernoulli type.*

$$\omega' - \alpha(q)\omega - \beta(q)\omega^3 = 0. \tag{11}$$

B. *Both functions α, β depend on p and a function $\omega_0(q) \not\equiv 0$ exists such that $\omega_0' - \alpha(p, q)\omega_0 - \beta(p, q)\omega_0^3 \equiv 0$, in which case $\omega = \pm\omega_0$.*

If $\theta = \theta(q)$ is a solution of (1) for a given Bernoulli function $q = q(p, \psi)$, $q_\psi \neq 0$, the flow defined in the physical plane by (4) is a GPM-flow for which the curve $h{:}\theta = \theta(q)$ in the hodograph plane is termed the *hodograph curve* of the flow.

The following theorem is a direct consequence of Lemma 1.

THEOREM 2. *The curve $h{:}\theta = \theta(q)$ will be the hodograph curve of a GPM-flow in the physical plane with a prescribed Bernoulli function $q(p, \psi)$, $q_\psi \neq 0$, $q_{pp} \not\equiv 0$, if, and only if, one of the following alternatives holds*

A. *The functions a, b, in* (9) *are functions of q alone, e.g., $a = a(q)$, $b = b(q)$ and $\theta(q)$ is any solution of the ordinary differential equation*

$$\theta'' - a(q)\theta' - b(q)\theta'^3 = 0. \tag{12}$$

B. *The functions, b in* (9) *are not functions of q alone and a function $\omega_0(q) \not\equiv 0$ exists such that*

$$\omega_0' - a(p, \psi)\omega_0 - b(p, \psi)\omega_0^3 \equiv 0,$$

in which case

$$\theta = \theta_0 \pm \int \omega_0(q)\, dq, \qquad \theta_0 = \text{const.} \tag{12'}$$

[17] This is certainly true as long as ω remains finite and different from zero.

In view of this theorem we distinguish two classes of GPM-flows of an abstract fluid for a prescribed Bernoulli function q with $q_\psi \not\equiv 0$, $q_{pp} \not\equiv 0$.

Class A. a, b in (9) *are functions of q alone, and if $a = a(q)$, $b = b(q)$, the hodograph curve h is a solution of* (12).

Class B. a, b, in (9) *are not functions of q alone but a function $\omega_0(q) \not\equiv 0$ exists such that $\omega_0' - a(p, \psi)\omega_0 - b(p, \psi)\omega_0^2 \equiv 0$, and the hodograph curve h is given by* (12′).

5. Isoenergetic, anisentropic GPM-flows. We restrict ourselves to a physical gas with a separable equation of state[18]

$$\rho = \Sigma(S)\cdot\Pi(p), \qquad p \geqq 0. \tag{13}$$

Consequently for the Bernoulli function q we find an equation of the form

$$q^2 = 2C(\psi) - \Psi(\psi)P(p), \quad \Psi(\psi) = 2\Sigma^{-1}(S(\psi)), \quad P(p) = \int_0^p \Pi^{-1}(p)\,dp. \tag{14}$$

In line with (6) we assume that Σ, Π are continuous functions meeting the following conditions

$$\text{(i)}\ 0 < \Sigma < +\infty, \quad \Pi(0) = 0, \quad 0 < \Pi < +\infty, \qquad \text{for } p > 0,$$

$$\text{(ii)}\ \Sigma' < 0 \quad \Pi' > 0, \qquad \text{for } p > 0$$

$$\text{(iii)}\ \int_0^p \Pi^{-1}\,dp < +\infty \qquad \text{for } p > 0.$$

Polytropic gases are examples of such gases for which

$$p = e^{-(S-S_0)/C_p}\cdot p^{1/\gamma}, \qquad C_p > 0, \qquad \gamma > 1, \qquad S_0 = \text{const.} \tag{15}$$

For isoenergetic flows we may, by a suitable choice of units, insure $C(\psi) = 1/2$ so that the Bernoulli function is given by

$$q^2 = 1 - \Psi(\psi)P(p), \tag{16}$$

where, from (7) we take Ψ, P to be continuous functions meeting the following conditions:

$$\text{(i)}\ 0 < \Psi < +\infty, \ \lim_{p\to 0} P' = +\infty; \ 0 < P' < +\infty \quad \text{for } 0 < p < p_0,$$

$$\text{(ii)}\ \Psi' \not\equiv 0; \quad P'' < 0 \qquad \text{for } 0 < p < p_0, \tag{17}$$

$$\text{(iii)}\ P(0) = 0, \quad 0 < P < \Psi^{-1} \qquad \text{for } 0 < p < p_0,$$

in which the stagnation pressure $p_0 = p_0(\psi)$ is, once $\Psi(\psi)$ is assigned, determined from

$$P(p_0) = \Psi^{-1}(\psi).$$

[18] The attention of the author was drawn to gases with a separable equation of state by R. C. Prim.

As an illustration, for a polytropic gas we have

$$P = \frac{p^{1-n}}{1-n}, \qquad 0 < n = \gamma^{-1} < 1, \tag{18}$$

and the stagnation pressure p_0 is

$$p_0 = \left(\frac{1-n}{\Psi(\psi)}\right)^{1/1-n}.$$

From (16) it is easy to verify that

$$\begin{aligned} q_p &= \tfrac{1}{2}(q - q^{-1})\frac{P'}{P}, \qquad q_{pp} = \tfrac{1}{2}(q - q^{-1})\left[\tfrac{1}{2}(1 + q^{-2}) - \left(\frac{P}{P'}\right)'\right]\left(\frac{P'}{P}\right)^2 \\ q_\psi &= \tfrac{1}{2}(q - q^{-1})\frac{\Psi'}{\Psi}, \qquad q_{p\psi} = \tfrac{1}{4}(q - q^{-1})(1 + q^{-2})\frac{\Psi'}{\Psi}\frac{P'}{P}, \\ & \qquad q_{\psi\psi} = \tfrac{1}{2}(q - q^{-1})\left[\tfrac{1}{2}(1 + q^{-2}) - \left(\frac{\Psi}{\Psi'}\right)'\right]\left(\frac{\Psi'}{\Psi}\right)^2. \end{aligned} \tag{19}$$

Since our attention is directed to GPM-flows, we calculate a, b in (9), to find

$$\begin{aligned} a &= 2(q - q^{-1})^{-1}\left(\frac{\Psi}{\Psi'}\right)' - q^{-3}\left\{\tfrac{1}{2}(1 + q^{-2}) - \left(\frac{P}{P'}\right)'\right\}^{-1}, \\ b &= \left\{\tfrac{1}{2}(q - q^{-1}) - q\left[\left(\frac{\Psi}{\Psi'}\right)' + \left(\frac{P}{P'}\right)'\right]\right\}\left\{\tfrac{1}{2}(1 + q^{-2}) - \left(\frac{P}{P'}\right)'\right\}^{-1}. \end{aligned} \tag{20}$$

For flows of Class A equations (20) represents two simultaneous equations for $(P/P')'$, $(\Psi/\Psi')'$ so that we can write

$$(P/P')' = F(q), \qquad (\Psi/\Psi')' = G(q).$$

One of F, G must be constant. Otherwise[19] q can be eliminated from these equations to obtain

$$(P/P')' = H((\Psi/\Psi')')$$

Since p, ψ are independent variables, this implies that both $(P/P')'$, $(\Psi/\Psi')'$ are constant, to contradict our assumption. Moreover, if either F or G is constant, the other is constant also. Indeed, from the first equation in (20) we have

$$F = (P/P')' = l = \text{const.}, \qquad (\Psi/\Psi')' = I(q),$$

or

$$G = (\Psi/\Psi')' = \lambda = \text{const.}, \qquad (P/P')' = J(q).$$

Under the first alternative $P \neq$ const. in (16). Since p, ψ are independent variables, q, ψ are also independent variables, and the second equation implies $(\Psi/\Psi')' = $ const. An analogous argument disposes of the second alternative.

We have seen that for flows of Class A

$$(P/P')' = l = \text{const.}; \qquad (\Psi/\Psi')' = \lambda = \text{const.},$$

[19] Here, and in what follows, tacit use is made of the assumption $q \neq$ const. made in the introduction.

necessarily. After integration one finds

$$(21)\quad \begin{array}{llll} \text{(a)}\ P = k(p - p_0)^{1/l}, & l \neq 0; & (\alpha)\ \Psi = [\kappa(\psi - \psi_0)]^{1/\lambda}, & \lambda \neq 0, \\ \text{(b)}\ P = e^{k(p-p_0)}, & l = 0; & (\beta)\ \Psi = e^{\kappa(\psi-\psi_0)}, & \lambda = 0, \end{array}$$

in which k, p_0, κ, ψ_0 denote integration constants. It follows from (17iii) that (21b) is impossible and that $p_0 = 0$, $l > 0$, $k > 0$ in (21a). Now (17ii) implies that actually $l > 1$ and with this it is easy to verify that conditions (17i) on P are also satisfied. On the other hand, Ψ can take either of the forms (21α), (21β) with $\kappa \neq 0$, provided $\psi - \psi_0$ is restricted, when necessary, so that $\Psi > 0$ in (21α). This leads to two alternative forms for the Bernoulli function

$$q^2 = 1 - k[\kappa(\psi - \psi_0)]^{1/\lambda} p^{1/l}, \qquad q^2 = 1 - ke^{\kappa(\psi-\psi_0)} p^{1/l}, \qquad l > 1,$$

corresponding to which the two alternative forms for the density functions are

$$(22)\qquad \rho = A[\kappa(\psi - \psi_0)]^{-1/\lambda} p^{1/\gamma}, \qquad \rho = Ae^{-\kappa(\psi-\psi_0)} p^{1/\gamma},$$

where

$$(23)\qquad A = 2l/k > 0, \qquad \gamma = l(l-1)^{-1} > 1.$$

and $\psi - \psi_0$ is subject to the condition $\rho > 0$ in the first form.

When (22), (23) are compared with (15) it appears that we are dealing with flows of a polytropic gas restricted to two alternative forms for the entropy distribution function, namely,

$$(24)\quad S(\psi) = \frac{C_p}{\lambda} \log|\psi - \psi_0| + \text{const.}, \quad S(\psi) = c(\psi - \psi_0) + \text{const.},$$

$$c = \text{const.} \neq 0,$$

and our results may be summed up in the following theorem.

THEOREM 3. *The only iso-energetic, anisentropic GPM-flows of Class A for a physical gas with a separable equation of state* (13) *are flows of polytropic gases with one of the entropy distribution functions* (24).

The next theorem completes this result.

THEOREM 4. *The only isoenergetic, anisentropic GMP-flows of a polytropic gas are of Class A, and the hodograph curves h constitute the solutions of the second order differential equation*

$$\theta'' - a(q)\theta' - b(q)\theta'^3 = 0,$$

with

$$a = 2\lambda(q - q^{-1})^{-1} - q^{-3}[\tfrac{1}{2}(1 + q^{-2}) - l]^{-1},$$

$$b = [\tfrac{1}{2}(q - q^{-1}) - (\lambda + l)q][\tfrac{1}{2}(1 + q^{-2}) - l]^{-1},$$

where $l = \gamma(\gamma - 1)^{-1}$ *and* λ *is a real parameter which serves to determine the entropy distribution function* $S(\psi)$ *as follows*:

$$S(\psi) = \frac{C_p}{\lambda} \log|\psi - \psi_0| + \text{const.}, \quad \lambda \neq 0; \quad S(\psi) = c(\psi - \psi_0) + \text{const.}, \quad \lambda = 0.$$

Here C_p is the specific heat at constant pressure and ψ_0, $c \neq 0$ are any real constants.

To prove this theorem we observe that for a polytropic gas $(P/P')' = (1 - n)^{-1} = l$ follows from (18) and (23). We place $(P/P')' = l$ in (20) and substitute for a, b in (8) from (20). This yields

$$\omega' - \frac{2}{q[(2l-1)q^2-1]}\omega - \frac{(2l-1)q^2+1}{(2l-1)q^2-1}q\omega^3 = -2q\omega\left[\frac{1}{1-q^2} - \frac{q^2\omega^2}{(2l-1)q^2-1}\right]\left(\frac{\Psi}{\Psi'}\right)'.$$

We have already excluded $q \equiv 0$ and $\omega \equiv 0$, and

$$\omega = \left(\frac{(2l-1)q^2-1}{q^2(1-q^2)}\right)^{1/2},$$

may also be excluded, since, as long as $(\Psi/\Psi')'$ is finite, ω cannot satisfy the above equation. We may accordingly solve for $(\Psi/\Psi')'$ obtaining

$$\left(\frac{\Psi}{\Psi'}\right)' = G(q),$$

which, since q, ψ are independent variables, implies $(\Psi/\Psi')' = \lambda = \text{const}$. Thus a, b in (20) must be the functions of q as stated in the theorem and the flow is of Class A. The rest of the theorem follows from (24).

It seems likely, therefore, that isoenergetic, anisentropic GPM-flows, of Class A are of major interest from the standpoint of applications, and we accordingly forego consideration of flows of Class B.

For non-isoenergetic, isentropic GPM-flows the following theorem holds.[20]

THEOREM 5. *There are non-isoenergetic, isentropic GPM-flows of Class A for a physical gas with a separable equation of state* (13).

6. Characteristics and hodograph curves. When the partial differentiations are carried out in (1), and this equation is regarded as a quasi-linear equation for θ, with q preassigned, it becomes

$$H\theta_{pp} + 2K\theta_{p\psi} + L\theta_{\psi\psi} + M = 0, \tag{25}$$

in which we have adopted the notation of Goursat[21] in setting

$$H = q\theta_\psi^2, \qquad K = -q\theta_p\theta_\psi, \qquad L = -(q_{pp} - q\theta_p^2), \tag{26}$$

$$M = (q_{p\psi} - q_\psi\theta_p^2 + 2q_p\theta_p\theta_\psi)\theta_\psi.$$

For a general flow $q \neq 0$, $\theta_\psi \not\equiv 0$ imply $H \not\equiv 0$ and, from (4), the relation

$$(q_{pp} - q\theta_p^2)^2 = (x_p^2 + y_p^2)\theta_\psi^2 = (dp/ds)^{-2}\theta_\psi^2$$

[20] For a proof of this Theorem see *Plane Rotational Prandtl-Meyer flows*. Naval Ordnance Laboratory Memorandum 10346, in particular §6.

[21] Goursat, E., *Leçons sur l'intégration des Équations aux dérivées du second ordre*. Paris (1896) vol. 1, pp. 46–47.

implies $L \not\equiv 0$, According to Goursat[21], the characteristic strips of (25) satisfy

$$H\,d\psi^2 - 2K\,dp\,d\psi + L\,dp^2 = 0, \qquad H\,d\xi\,d\psi + L\,d\eta\,dp + M\,dp\,d\psi = 0. \tag{27}$$

Since the variables p, ω, θ have taken over the roles played by the variables x, y, z of Goursat, the notation differs from his, the connection between the two notations being illustrated in the table below

$$x, y, z, p, q,$$
$$p, \psi, \theta, \xi, \eta.$$

When we substitute from (26) into the first equatino of (27), this equation reduces to

$$(\theta_p \pm \sqrt{q_{pp}/q})\,dp/d\psi + \theta_\psi = 0. \tag{28}$$

Let $q(p, \psi)$ be the prescribed Bernoulli function and $\theta(p, \psi)$ be a solution of (25). The region R_0 of the (p, ψ)-plane characterized by $q_{pp} > 0$ is carried by (4) into a region R of the physical plane within which the flow is supersonic[22]. If we substitute $\theta(p, \psi)$ for θ in (28), the solution curves of (28) form a curvilinear network in R_0, two curves of the network passing in general through each point of R_0. The curves in this network constitute the *characteristic base curves*[23] of the solution $\theta = \theta(p, \psi)$. These characteristic base curves are carried by (5), i.e. by

$$u = q(p, \psi)\cos\theta(p, \psi), \qquad v = q(p, \psi)\sin\theta(p, \psi),$$

into a network of curves in the hodograph plane which we term characteristics.

In general the characteristics depend on the solution $\theta(p, \psi)$, i.e., for a given Bernoulli function $q(p, \psi)$, the characteristics in the hodograph plane are not the same for all flows in the physical plane, but vary from one flow to another.

Irrotational flows have the remarkable property that, once the Bernoulli function $q = q(p)$ is assigned, the characteristics in the hodograph plane are the same for all flows in the physical plane. This is immediate when we write (28) in the form $d\theta = \pm\sqrt{q_{pp}/q}\,dp$. Since $q = q(p)$, we may integrate and the characteristics are presented in parametric form in the hodograph plane by

$$q = q(p), \qquad \theta = \pm\int \sqrt{q_{pp}/q}\,dq + \theta_0. \tag{29}$$

In the case of a polytropic gas the Bernoulli function is, from (16), (18), given by

$$q^2 = 1 - Ep^{1-\gamma^{-1}}, \qquad \gamma > 1, \qquad E > 0, \tag{30}$$

the constant E, determined by the entropy, serving to fix the Bernoulli function.

[21] See Theorem 3 in 5.

[23] Courant, R., Hilbert, D., Methoden der Mathematischen Physik II, Berlin (1937) p. 292.

It turns out that the hodograph curves for irrotational PM-flows having the above Bernoulli function $q = q(p)$ are precisely[24] the characteristics (29).

The situation is entirely different for rotational flows. First of all, as has been indicated above, the characteristics in the hodograph plane need not be the same for all flows in the physical plane having the same Bernoulli function. Secondly, instead of the two one-parameter families (29) of hodograph curves for PM-flows with a given Bernoulli function, it is possible to have a two parameter family. For example, in isoenergetic, anisentropic GPM-flows of a polytropic gas (30) may be replaced by

$$q^2 = 1 - k[\kappa(\psi - \psi_0)]^{1/\lambda} p^{1-(1/\gamma)}, \qquad \lambda \neq 0, \qquad k > 0, \tag{31}$$

and θ taken to be any solution of the differential equation of second order in Theorem 4. Finally, in place of the one-parameter family (30) of Bernoulli functions for PM-flows of a given polytropic gas (γ fixed), (31) offers a two-parameter family of Bernoulli functions for PM-flows of the same gas.

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[24] See §5 in 3.

ON A VARIATIONAL THEOREM IN ELASTICITY*

By Eric Reissner

1. Introduction. There are in the theory of elasticity two well known variational theorems or principles, one of them being for *displacements* and the other one being for *stresses* [1]. The former is also called the principle of minimum potential energy while the latter is often referred to as the principle of minimum complementary energy. With reference to the differential equations of the theory of elasticity, which consist of equations of equilibrium for the components of stress and of the stress-displacement relations, a possible characterisation of the difference between the two variational theorems is as follows. In the theorem for displacements the stress-displacement relations are taken as equations of definition for the components of stress in terms of appropriate displacement derivatives and the variational equation is equivalent to the system of differential equations of equilibrium. In the theorem for stresses the differential equations of equilibrium serve to restrict the class of admissible stress variations and the variational equation is equivalent to the system of stress-displacement relations.

Both variational principles have been found valuable for the purpose of obtaining approximate solutions of boundary value problems. When viewed in the above light these approximate solutions are such that *part* of the complete system of differential equations (either the stress-displacement relations or the equations of equilibrium) is satisfied exactly while the remaining equations are satisfied approximately only.

It is natural to ask whether it might not be possible to use the calculus of variations for the purpose of obtaining approximate solutions in such a manner that there is no preferential treatment for either one of the two kinds of differential equations which occur in the theory. In what follows this question is answered in the affirmative. A variational problem will be formulated which has both the equation of equilibrium *and* the stress-displacement relations as appropriate Euler equations.

It is also shown that application of the theorem to a problem which had previously been treated by means of the method of complementary energy leads to the results obtained by the complementary-energy method in a manner which represents some simplification over the earlier derivations.

2. The boundary value problem. Let σ_x, τ_{xy}, etc. be components of stress in the usual notation and let u, v, w be components of displacement. Define quantities ϵ_x, γ_{xy}, etc. in the usual way in terms of displacements by $\epsilon_x = \partial u/\partial x$, $\gamma_{xy} = \partial u/\partial y + \partial v/\partial x$, etc.

The differential equations to be considered are the three equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \text{etc.} \tag{1}$$

* The results presented here were obtained in the course of studies supported by the Office of Naval Research under Contract N5ori-07834.

(where for simplicity's sake body forces are not considered), and the six stress-displacement relations

$$\epsilon_x = \frac{\partial W}{\partial \sigma_x}, \qquad \gamma_{xy} = \frac{\partial W}{\partial \tau_{xy}}, \qquad \text{etc.} \tag{2}$$

In Eqs. (2) W is a given function of the six arguments σ_x, τ_{xy}, etc.

Let

$$p_x = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, z) \tag{3}$$

be the x-component of the stresses acting on an element of area with normal direction n, with corresponding definitions for p_y and p_z.

The system (1) and (2) is to be solved for a region bounded by a surface S subject to the following boundary conditions on S,

$$\left.\begin{aligned} p_x &= \bar{p}_x \quad \text{or } u = \bar{u} \\ p_y &= \bar{p}_y \quad \text{or } v = \bar{v} \\ p_z &= \bar{p}_z \quad \text{or } w = \bar{w} \end{aligned}\right\} \tag{4}$$

A bar indicates a given function and equations (4) are to be understood in the sense that on that part of S where p_x is prescribed u is not prescribed while on the remaining part of the surface the reverse is true. Corresponding statements hold for the other two formulas in (4).

3. The variational theorem. We introduce a function F which is defined in terms of twelve arguments σ_x, τ_{xy}, $\cdots$, ϵ_x, γ_{xy}, $\cdots$ by the relation

$$F = \sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} - W \tag{5}$$

The variational theorem may be stated in the following form.

Among all states of stress and displacement which satisfy the boundary conditions of prescribed surface displacement the actually occurring state of stress and displacement is determined by the variational equation

$$\delta\left\{\iiint_V F\, dV - \iint_{S_1} (\bar{p}_x u + \bar{p}_y v + \bar{p}_z w)\, dS\right\} = 0. \tag{6}$$

The symbol S_1 in equation (6) indicates that each of the three surface integrals is to be taken over that part of the surface only where the appropriate surface stress is prescribed.

For the proof of the theorem it must be shown that the Euler equations of the variational problem (6) are the differential equations (1) and (2) and that the natural boundary conditions corresponding to (6) are equations (4).

Writing equation (6) in the form

$$\delta\left\{\iiint_V \left[\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \cdots - W(\sigma_x, \cdots)\right] dV - \iint_{S_1} [\bar{p}_x u + \bar{p}_y v + \bar{p}_z w]\, dS\right\} = 0 \tag{7}$$

there follows first

$$\iiint_V \left[\left(\frac{\partial u}{\partial x} - \frac{\partial W}{\partial \sigma_x} \right) \delta\sigma_x + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial W}{\partial \tau_{xy}} \right) \delta\tau_{xy} + \cdots \right.$$
$$\left. + \sigma_x \frac{\partial \delta u}{\partial x} + \tau_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) + \cdots \right] dV$$
$$- \iint_{S_1} [\, \bar{p}_x \delta u + \bar{p}_y \delta v + \bar{p}_z \delta w] \, dS = 0 \qquad (8)$$

Elimination in (8) of partial derivatives of displacement variations by integration by parts leads to the final form of the variational equation,

$$\iiint_V \left[\left(\frac{\partial u}{\partial x} - \frac{\partial W}{\partial \sigma_x} \right) \delta\sigma_x + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \delta\tau_{xy} + \cdots \right.$$
$$\left. - \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta u - \cdots \right] dV$$
$$+ \iint_{S_1} [(p_x - \bar{p}_x)\delta u + \cdots] \, dS + \iint_{S_2} [p_x \delta u + \cdots] \, dS = 0 \qquad (9)$$

where S_2 is the part of S which does not belong to S_1.

The integrals over S_2 vanish because in them $\delta u = 0$, etc. The vanishing of the volume integral requires satisfaction of the equilibrium equations (1) and of the stress-displacement relations (2) and the vanishing of the surface integrals over S_1 requires satisfaction of the stress boundary conditions. It has thus been shown that the variational equation (6), under the assumed restrictions concerning variations of boundary displacements, is equivalent to the complete system of the differential equations (1) and (2) and of the boundary conditions (4). This is the result which was to be proved.

We may note the fact that extensions of the theorem are possible to more general systems of boundary conditions, to thermo-elastic and to dynamic problems, and to problems of finite strain.

4. An application of the variational theorem. As an example of application consider the problem of deriving a system of two-dimensional equations for transverse bending of plates. Let $z = \pm h/2$ be the equations of the faces of the plate. Assume a distribution $p(x, y)$ of normal stress and no tangential stress over the face $z = h/2$ and let the face $z = -h/2$ be unloaded. For simplicity sake assume further that all boundary conditions at the cylindrical boundary $f(x, y) = 0$ of the plate are displacement boundary conditions.

For an isotropic plate material which obeys Hooke's Law the variational equation (6) reads then as follows

$$\delta \left\{ \iiint_{-h/2}^{h/2} \left[\sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \cdots \right. \right.$$
$$\left. \left. - \frac{1}{2E} \left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) \right. \right. \right.$$

$$+ 2(1+\nu)(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\}\Big] dz\,dx\,dy$$

$$- \iint p w(x, y, h/2)\,dx\,dy\Big\} = 0 \tag{10}$$

In order to obtain a two-dimensional system of equations the following expressions are used as approximations to the true stresses

$$\begin{aligned}
\sigma_x &= \frac{M_x}{h^3/6}\frac{z}{h/2}, \qquad \sigma_y = \frac{M_y}{h^3/6}\frac{z}{h/2}, \qquad \tau_{xy} = \frac{H}{h^3/6}\frac{z}{h/2} \\
\tau_{xz} &= \frac{V_x}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \qquad \tau_{yz} = \frac{V_y}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \\
\sigma_z &= \frac{3p}{4}\left[\frac{2}{3} + \frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right]
\end{aligned} \tag{11}$$

Equations (11) are of the same form as those used in an earlier consideration of the same problem by means of the method of complementary energy [2]. Substitution of (11) into the variational equation (10) permits explicit integration with respect to z in all terms which are quadratic in the stresses and suggests introduction of the following weighted displacement averages

$$\left.\begin{aligned}
\alpha &= \frac{12}{h^3}\int_{-h/2}^{h/2} u\,z\,dz, \qquad \beta = \frac{12}{h^3}\int_{-h/2}^{h/2} v\,z\,dz \\
w_0 &= \frac{3}{2h}\int_{-h/2}^{h/2} w\left[1 - \left(\frac{z}{h/2}\right)^2\right] dz
\end{aligned}\right\} \tag{12}$$

The resulting variational equation is of the form

$$\begin{aligned}
\delta \iint \Big\{\Big[& M_x\frac{\partial\alpha}{\partial x} + H\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) + M_y\frac{\partial\beta}{\partial y} + V_x\left(\alpha + \frac{\partial w_0}{\partial x}\right) \\
& + V_y\left(\beta + \frac{\partial w_0}{\partial y}\right) - p w_0\Big] - \frac{1}{2E}\Big[\frac{12}{h^3}(M_x^2 + M_y^2 - 2\nu M_x M_y \\
& + 2(1+\nu)H^2) + \frac{12}{5h^3}(1+\nu)(V_x^2 + V_y^2) - \frac{12\nu}{5h}(M_x + M_y)p\Big] dx\,dy = 0
\end{aligned} \tag{13}$$

After carrying out the variations, integrating by parts and taking account of the fact that for the assumed boundary conditions the variations of the displacement averages vanish at the boundary, the following equation is obtained

$$\begin{aligned}
\iint \Big\{ & \Big[V_x - \frac{\partial M_x}{\partial x} - \frac{\partial H}{\partial y}\Big]\delta\alpha + \Big[V_y - \frac{\partial H}{\partial x} - \frac{\partial M_y}{\partial y}\Big]\delta\beta - \Big[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p\Big]\delta w_0 \\
& + \Big[\frac{\partial\alpha}{\partial x} - \frac{12}{Eh^3}(M_x - \nu M_y) + \frac{6\nu}{5Eh}p\Big]\delta M_x \\
& + \Big[\frac{\partial\beta}{\partial y} - \frac{12}{Eh^3}(M_y - \nu M_x) + \frac{6\nu}{5Eh}p\Big]\delta M_y + \Big[\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x} - \frac{12}{Gh^3}H\Big]\delta H \\
& + \Big[\alpha + \frac{\partial w_0}{\partial x} - \frac{6}{5Gh}V_x\Big]\delta V_x + \Big[\beta + \frac{\partial w_0}{\partial y} - \frac{6}{5Gh}V_y\Big]\delta V_y\Big\}\,dx\,dy = 0
\end{aligned} \tag{14}$$

The contents of every bracket in (14) must vanish. The first three of the resulting equations are, as was to be expected, the differential equations of equilibrium of plate theory. The remaining equations are the appropriate stress-displacement relations of plate theory when deformation due to transverse shearing and normal stresses are taken into account [2].

In comparing the present with the earlier derivation of the same results the fact that the equilibrium equations are now obtained from the same variational equation as the stress-displacement relations is of little importance for the present problem since direct derivation of the equilibrium equations is simple. A considerable simplification may, however, be thought to be the fact that the present derivation avoids the use of the Lagrangian multiplier method which was basic in the earlier work and which made necessary special considerations in order to determine the physical significance of these multipliers.

The foregoing derivation has in common with the derivation by means of the method of complementary energy the consequence that approximations are obtained for weighted averages of the displacements rather than for the displacements themselves. In order to obtain approximations for the displacements themselves one may use equations (12) and in them introduce the approximations

$$u = \alpha^* z, \qquad v = \beta^* z, \qquad w = w_0^*(x, y) \tag{15}$$

If this is done it is found that $\alpha^* = \alpha$, $\beta^* = \beta$ and $w_0^* = w_0$.

In contrast to this it is found that if the problem is analysed by means of the method of potential energy, approximations such as equations (15) may be made for displacements and the energy method furnishes equations for weighted averages of the stresses [3]. In view of the fact that the various terms in the expressions for the strain energy are obtained in the potential energy method by differentiations of expressions such as (15) while no such differentiations are required with the use of the complementary energy method it is likely that the complementary energy method results in a more accurate system of equations than does the potential energy method.

In addition to this a certain difficulty appears in the use of the potential energy method for the problem of plate bending in connection with the approximation $w = w_0^*(x, y)$. It is found that in order to be able to use this approximation the further unrelated and formally incompatible assumption must be made that in the general expression for the strain energy the terms with ϵ_z are first eliminated by setting in the stress strain relations $\sigma_z = 0$, making for an isotropic medium $\epsilon_z = -[\nu/(1 - \nu)](\epsilon_x + \epsilon_y)$.

No such difficulty occurs in the application of the present variational theorem. It is in fact possible to use *simultaneously* in the variational equation (10) the approximations (11) for the stresses and the approximations (15) for the displacements. The result is a variational equation of the same form as (14) except that in it α, β and w_0 are replaced by the corresponding starred quantities. This apparent greater flexibility of the variational theorem obtained here may possibly be found useful in connection with still other problems of the theory of elasticity.

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COEFFICIENTS FOR POLAR COMPLEX INTERPOLATION

By Herbert E. Salzer

In two previous articles by the author[I, II], formulas for Lagrangian interpolation for complex analytic functions tabulated around a circular arc at points differing in phase by the constant amount θ, are given as trigonometric functions of the angle θ. The present paper supplements the preceding ones by giving the numerical values of those functions of θ, for six cases quite likely to arise in practise, namely $\theta = 1°, 5°, 10°, 15°, 20°$ and $30°$. Paper I gave the Lagrangian coefficients themselves, up to the 5-point case, whereas paper II gave auxiliary coefficients for Lagrangian interpolation up to the 9-point case. Here Part I supplements paper I and Part II supplements paper II. In Part I all coefficients are given to eight decimal places. In Part II all coefficients are given to nine decimal places of which the first eight are guaranteed, while the last place is probably correct, but not guaranteed.

For the meaning and use of these coefficients, the reader is referred to those two above-mentioned papers I and II. To avoid any confusion, exactly the same notation is kept for the numerical coefficients, as appears in the formulas in those earlier articles. The only trivial change has been in Part II, where for most values of θ and n, instead of tabulating the quantities $A_k^{(n)}$ as originally defined, there are tabulated $A_k^{(n)}$ multiplied by a suitable power of 10, while keeping the same notation $A_k^{(n)}$. This does not affect the procedure in using $A_k^{(n)}$ (owing to the fact that the interpolation formula (2) in paper II is unchanged when $A_k^{(n)}$ is replaced by $CA_k^{(n)}$), and it eliminates the large number of zeros that would have to be retained needlessly.

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I "Formulas for Direct and Inverse Interpolation of a Complex Function Tabulated Along Equidistant Circular Arcs," J. Math. and Phys., Nov. 1945, pp. 141–143.

II "Alternative Formulas for Direct Interpolation of a Complex Function Tabulated Along Equidistant Circular Arcs," J. Math. and Phys., Apr. 1947, pp. 56–61.

PART I

$\theta = 1°$

Three-Point

Real L_{-1}	=	$+.49984\ 770p_2$	$-.01308\ 897q_2$	
		$-.49984\ 770p_1$	$+.01308\ 897q_1$	
Real L_0	=	$-.99984\ 770p_2$	$+.01745\ 241q_2$	
		$-.00015\ 230p_1$	$-.01745\ 241q_1$	$+1$
Real L_1	=	$+.50000\ 000p_2$	$-.00436\ 343q_2$	
		$+.50000\ 000p_1$	$+.00436\ 343q_1$	

PART I—*Continued*

Four-Point

Real L_{-1} = $-.16663\ 282p_3$ $+.00436\ 343q_3$ $+.49979\ 692p_2$ $-.01599\ 778q_2$
$-.33316\ 410p_1$ $+.01163\ 435q_1$

Real L_0 = $+.50000\ 000p_3$ $-.00436\ 343q_3$ $-.99984\ 770p_2$ $+.02617\ 927q_2$
$-.50015\ 230p_1$ $-.02181\ 584q_1$ $+1$

Real L_1 = $-.50000\ 000p_3$ $-.00436\ 343q_3$ $+.50015\ 230p_2$ $-.01308\ 897q_2$
$+.99984\ 770p_1$ $+.01745\ 241q_1$

Real L_2 = $+.16663\ 282p_3$ $+.00436\ 343q_3$ $-.00010\ 153p_2$ $+.00290\ 748q_2$
$-.16653\ 129p_1$ $-.00727\ 091q_1$

Five-Point

Real L_{-2} = $+.04152\ 181p_4$ $-.00363\ 269q_4$ $-.08291\ 683p_3$ $+.00871\ 469q_3$
$-.04170\ 569p_2$ $+.00145\ 817q_2$ $+.08310\ 070p_1$ $-.00654\ 017q_1$

Real L_{-1} = $-.16637\ 903p_4$ $+.01017\ 618q_4$ $+.16577\ 004p_3$ $-.02179\ 635q_3$
$+.66668\ 341p_2$ $-.01746\ 126q_2$ $-.66607\ 442p_1$ $+.02908\ 144q_1$

Real L_0 = $+.24986\ 673p_4$ $-.00872\ 554q_4$ $+.00057\ 109p_3$ $+.02180\ 920q_3$
$-1.25013\ 322p_2$ $+.02182\ 116q_2$ $-.00030\ 461p_1$ $-.03490\ 481q_1+1$

Real L_1 = $-.16668\ 359p_4$ $+.00145\ 463q_4$ $-.16668\ 359p_3$ $-.01309\ 074q_3$
$+.66688\ 667p_2$ $-.00581\ 629q_2$ $+.66648\ 051p_1$ $+.01745\ 241q_1$

Real L_2 = $+.04167\ 407p_4$ $+.00072\ 742q_4$ $+.08325\ 928p_3$ $+.00436\ 321q_3$
$-.04173\ 117p_2$ $-.00000\ 177q_2$ $-.08320\ 219p_1$ $-.00508\ 886q_1$

Note: For Imaginary L_j, replace p_i by q_i, and q_i by $(-p_i)$.

$\theta = 5°$

Three-Point

Real L_{-1} = $+.49619\ 470p_2$ $-.06532\ 527q_2$
$-.49619\ 470p_1$ $+.06532\ 527q_1$

Real L_0 = $-.99619\ 470p_2$ $+.08715\ 574q_2$
$-.00380\ 530p_1$ $-.08715\ 574q_1$ $+1$

Real L_1 = $+.50000\ 000p_2$ $-.02183\ 047q_2$
$+.50000\ 000p_1$ $+.02183\ 047q_1$

Four-Point

Real L_{-1} = $-.16581\ 889p_3$ $+.02183\ 047q_3$ $+.49492\ 304p_2$ $-.07986\ 041q_2$
$-.32910\ 414p_1$ $+.05802\ 994q_1$

Real L_0 = $+.50000\ 000p_3$ $-.02183\ 047q_3$ $-.99619\ 470p_2$ $+.13081\ 669q_2$
$-.50380\ 530p_1$ $-.10898\ 621q_1$ $+1$

Real L_1 = $-.50000\ 000p_3$ $-.02183\ 047q_3$ $+.50380\ 530p_2$ $-.06532\ 527q_2$
$+.99619\ 470p_1$ $+.08715\ 574q_1$

PART I—*Continued*

Four-Point (*cont.*)

Real L_2 = $+.16581\ 889p_2$ $+.02183\ 047q_2$ $-.00253\ 364p_2$ $+.01436\ 900q_2$
$-.16328\ 525p_1$ $-.03619\ 947q_1$

Five-Point

Real L_{-2} = $+.03807\ 593p_4$ $-.01775\ 510q_4$ $-.07305\ 694p_3$ $+.04214\ 726q_3$
$-.04257\ 341p_2$ $+.00773\ 193q_2$ $+.07755\ 442p_1$ $-.03212\ 409q_1$

Real L_{-1} = $-.15950\ 891p_4$ $+.05029\ 297q_4$ $+.14443\ 229p_3$ $-.10655\ 970q_3$
$+.66697\ 493p_2$ $-.08825\ 565q_3$ $-.65189\ 831p_1$ $+.14452\ 238q_1$

Real L_0 = $+.24667\ 127p_4$ $-.04349\ 480q_4$ $+.01423\ 911p_3$ $+.10815\ 676q_3$
$-1.25329\ 977p_2$ $+.10964\ 952q_3$ $-.00761\ 060p_1$ $-.17431\ 149q_1+1$

Real L_1 = $-.16709\ 055p_4$ $+.00729\ 533q_4$ $-.16709\ 055p_3$ $-.06554\ 694q_3$
$+.67216\ 752p_2$ $-.02890\ 414q_3$ $+.66201\ 359p_1$ $+.08715\ 574q_1$

Real L_2 = $+.04185\ 227p_4$ $+.00366\ 160q_4$ $+.08147\ 610p_3$ $+.02180\ 260q_3$
$-.04326\ 926p_2$ $-.00022\ 166q_3$ $-.08005\ 910p_1$ $-.02524\ 254q_1$

$\theta = 10°$

Three-Point

Real L_{-1} = $+.48480\ 775p_2$ $-.12990\ 385q_2$
$-.48480\ 775p_1$ $+.12990\ 385q_1$

Real L_0 = $-.98480\ 775p_2$ $+.17364\ 818q_2$
$-.01519\ 225p_1$ $-.17364\ 818q_1$ $+1$

Real L_1 = $+.50000\ 000p_2$ $-.04374\ 433q_2$
$+.50000\ 000p_1$ $+.04374\ 433q_1$

Four-Point

Real L_{-1} = $-.16325\ 607p_3$ $+.04374\ 433q_3$ $+.47969\ 186p_2$ $-.15891\ 754q_2$
$-.31643\ 579p_1$ $+.11517\ 321q_1$

Real L_0 = $+.50000\ 000p_3$ $-.04374\ 433q_3$ $-.98480\ 775p_2$ $+.26113\ 684q_2$
$-.51519\ 225p_1$ $-.21739\ 251q_1$ $+1$

Real L_1 = $-.50000\ 000p_3$ $-.04374\ 433q_3$ $+.51519\ 225p_2$ $-.12990\ 385q_2$
$+.98480\ 775p_1$ $+.17364\ 818q_1$

Real L_2 = $+.16325\ 607p_3$ $+.04374\ 433q_3$ $-.01007\ 635p_2$ $+.02768\ 454q_2$
$-.15317\ 972p_1$ $-.07142\ 888q_1$

Five-Point

Real L_{-2} = $+.02768\ 454p_4$ $-.03299\ 315q_4$ $-.04391\ 069p_3$ $+.07560\ 105q_3$
$-.04445\ 156p_2$ $+.01806\ 980q_2$ $+.06067\ 770p_1$ $-.06067\ 770q_1$

Real L_{-1} = $-.13844\ 908p_4$ $+.09694\ 309q_4$ $+.07997\ 411p_3$ $-.19822\ 471q_3$
$+.66653\ 956p_2$ $-.18226\ 355q_3$ $-.60806\ 459p_1$ $+.28354\ 517q_1$

Real L_0 = $+.23672\ 132p_4$ $-.08615\ 951q_4$ $+.05648\ 025p_3$ $+.21078\ 715q_3$
$-1.26281\ 707p_2$ $+.22266\ 872q_3$ $-.03038\ 449p_1$ $-.34729\ 636q_1+1$

PART I—*Continued*

Real L_1 = $-.16837\ 197p_4$ $+.01473\ 064q_4$ $-.16837\ 197p_3$ $-.13168\ 058q_3$
$+.68868\ 011p_2$ $-.05669\ 824q_2$ $+.64806\ 382p_1$ $+.17364\ 818q_1$

Real L_2 = $+.04241\ 518p_4$ $+.00747\ 894q_4$ $+.07582\ 830p_3$ $+.04351\ 709q_3$
$-.04795\ 104p_2$ $-.00177\ 673q_2$ $-.07029\ 244p_1$ $-.04921\ 930q_1$

$\theta = 15°$

Three-Point

Real L_{-1} = $+.46592\ 583p_2$ $-.19299\ 280q_2$
$-.46592\ 583p_1$ $+.19299\ 280q_1$

Real L_0 = $-.96592\ 583p_2$ $+.25881\ 905q_2$
$-.03407\ 417p_1$ $-.25881\ 905q_1$ $+1$

Real L_1 = $+.50000\ 000p_2$ $-.06582\ 625q_2$
$+.50000\ 000p_1$ $+.06582\ 625q_1$

Four-Point

Real L_{-1} = $-.15891\ 862p_3$ $+.06582\ 625q_3$ $+.45430\ 376p_2$ $-.23636\ 694q_2$
$-.29538\ 514p_1$ $+.17054\ 069q_1$

Real L_0 = $+.50000\ 000p_3$ $-.06582\ 625q_3$ $-.96592\ 583p_2$ $+.39047\ 154q_2$
$-.53407\ 417p_1$ $-.32464\ 529q_1$ $+1$

Real L_1 = $-.50000\ 000p_3$ $-.06582\ 625q_3$ $+.53407\ 417p_2$ $-.19299\ 280q_2$
$+.96592\ 583p_1$ $+.25881\ 905q_1$

Real L_2 = $+.15891\ 862p_3$ $+.06582\ 625q_3$ $-.02245\ 211p_2$ $+.03888\ 819q_2$
$-.13646\ 652p_1$ $-.10471\ 444q_1$

Five-Point

Real L_{-2} = $+.01162\ 207p_4$ $-.04337\ 414q_4$ $-.00079\ 202p_3$ $+.09276\ 431q_3$
$-.04490\ 422p_2$ $+.03287\ 217q_2$ $+.03407\ 417p_1$ $-.08226\ 233q_1$

Real L_{-1} = $-.10471\ 444p_4$ $+.13646\ 652q_4$ $-.02013\ 001p_3$ $-.26131\ 096q_3$
$+.66141\ 054p_2$ $-.28687\ 719q_2$ $-.53656\ 609p_1$ $+.41172\ 164q_1$

Real L_0 = $+.22025\ 892p_4$ $-.12716\ 655q_4$ $+.12530\ 841p_3$ $+.30252\ 125q_3$
$-1.27741\ 898p_2$ $+.34228\ 338q_2$ $-.06814\ 835p_1$ $-.51763\ 809q_1+1$

Real L_1 = $-.17054\ 069p_4$ $+.02245\ 211q_4$ $-.17054\ 069p_3$ $-.19900\ 882q_3$
$+.71623\ 693p_2$ $-.08226\ 233q_2$ $+.62484\ 445p_1$ $+.25881\ 905q_1$

Real L_2 = $+.04337\ 414p_4$ $+.01162\ 207q_4$ $+.06615\ 432p_3$ $+.06503\ 422q_3$
$-.05532\ 427p_2$ $-.00601\ 602q_2$ $-.05420\ 418p_1$ $-.07064\ 027q_1$

$\theta = 20°$

Three-Point

Real L_{-1} = $+.43969\ 262p_2$ $-.25385\ 665q_2$
$-.43969\ 262p_1$ $+.25385\ 665q_1$

Real L_0 = $-.93969\ 262p_2$ $+.34202\ 014q_2$
$-.06030\ 738p_1$ $-.34202\ 014q_1$ $+1$

PART 1—*Continued*

Three-Point (*cont.*)

Real L_1 = $+.50000\ 000p_3 - .08816\ 349q_3$
$+.50000\ 000p_1 + .08816\ 349q_1$

Four-Point

Real L_{-1} = $-.15270\ 364p_3 + .08816\ 349q_3 + .41874\ 809p_3 - .31140\ 128q_3$
$-.26604\ 444p_1 + .22323\ 779q_1$

Real L_0 = $+.50000\ 000p_3 - .08816\ 349q_3 - .93969\ 262p_3 + .51834\ 712q_3$
$-.56030\ 738p_1 - .43018\ 363q_1 + 1$

Real L_1 = $-.50000\ 000p_3 - .08816\ 349q_3 + .56030\ 738p_3 - .25385\ 665q_3$
$+.93969\ 262p_1 + .34202\ 014q_1$

Real L_2 = $+.15270\ 364p_3 + .08816\ 349q_3 - .03936\ 285p_3 + .04691\ 081q_3$
$-.11334\ 080p_1 - .13507\ 430q_1$

Five-Point

Real L_{-2} = $-.00827\ 164p_4 - .04691\ 081q_4 + .04863\ 217p_3 + .08816\ 349q_3$
$-.04036\ 053p_2 + .05256\ 895q_2 + .00000\ 000p_1 - .09382\ 163q_1$

Real L_{-1} = $-.06030\ 738p_4 + .16569\ 316q_4 - .14542\ 968p_3 - .28447\ 551q_3$
$+.64542\ 968p_2 - .40522\ 291q_2 - .43969\ 262p_1 + .52400\ 526q_1$

Real L_0 = $+.19746\ 542p_4 - .16569\ 316q_4 + .21840\ 995p_3 + .37829\ 714q_3$
$-1.29526\ 062p_2 + .47143\ 631q_2 - .12061\ 476p_1 - .68404\ 029q_1 + 1$

Real L_1 = $-.17364\ 818p_4 + .03061\ 886q_4 - .17364\ 818p_3 - .26818\ 356q_3$
$+.75490\ 009p_2 - .10445\ 544q_2 + .59239\ 627p_1 + .34202\ 014q_1$

Real L_2 = $+.04476\ 178p_4 + .01629\ 195q_4 + .05203\ 574p_3 + .08619\ 844q_3$
$-.06470\ 863p_2 - .01432\ 690q_2 - .03208\ 889p_1 - .08816\ 349q_1$

$\theta = 30°$

Three-Point

Real L_{-1} = $+.36602\ 540p_3 - .36602\ 540q_3$
$-.36602\ 540p_1 + .36602\ 540q_1$

Real L_0 = $-.86602\ 540p_3 + .50000\ 000q_3$
$-.13397\ 460p_1 - .50000\ 000q_1 + 1$

Real L_1 = $+.50000\ 000p_3 - .13397\ 460q_3$
$+.50000\ 000p_1 + .13397\ 460q_1$

Four-Point

Real L_{-1} = $-.13397\ 460p_3 + .13397\ 460q_3 + .31698\ 730p_3 - .45096\ 189q_3$
$-.18301\ 270p_1 + .31698\ 730q_1$

Real L_0 = $+.50000\ 000p_3 - .13397\ 460q_3 - .86602\ 540p_3 + .76794\ 919q_3$
$-.63397\ 460p_1 - .63397\ 460q_1 + 1$

Real L_1 = $-.50000\ 000p_3 - .13397\ 460q_3 + .63397\ 460p_3 - .36602\ 540q_3$
$+.86602\ 540p_1 + .50000\ 000q_1$

PART 1—*Concluded*

Real L_2	=	$+.13397\ 460p_3$	$+.13397\ 460q_3$	$-.08493\ 649p_2$	$+.04903\ 811q_2$
		$-.04903\ 811p_1$	$-.18301\ 270q_1$		
			Five-Point		
Real L_{-2}	=	$-.04903\ 811p_4$	$-.02831\ 216q_4$	$+.12638\ 837p_3$	$+.00758\ 622q_3$
		$+.00000\ 000p_2$	$+.09807\ 621q_2$	$-.07735\ 027p_1$	$-.07735\ 027q_1$
Real L_{-1}	=	$+.04903\ 811p_4$	$+.18301\ 270q_4$	$-.41506\ 351p_3$	$-.18301\ 270q_3$
		$+.54903\ 811p_2$	$-.68301\ 270q_3$	$-.18301\ 270p_1$	$+.68301\ 270q_1$
Real L_0	=	$+.13397\ 460p_4$	$-.23205\ 081q_4$	$+.46410\ 162p_3$	$+.46410\ 162q_3$
		$-1.33012\ 702p_2$	$+.76794\ 919q_2$	$-.26794\ 919p_1$	$-1.00000\ 000q_1+1$
Real L_1	=	$-.18301\ 270p_4$	$+.04903\ 811q_4$	$-.18301\ 270p_3$	$-.41506\ 351q_3$
		$+.86602\ 540p_2$	$-.13397\ 460q_3$	$+.50000\ 000p_1$	$+.50000\ 000q_1$
Real L_2	=	$+.04903\ 811p_4$	$+.02831\ 216q_4$	$+.00758\ 622p_3$	$+.12638\ 837q_3$
		$-.08493\ 649p_2$	$-.04903\ 811q_3$	$+.02831\ 216p_1$	$-.10566\ 243q_1$

PART II

$\theta = 1°$		$\theta = 5°$	
Three-Point		**Three-Point**	
$A_{-1}^{(3)}$	$-.174524064 - .001523048\ i$	$A_{-1}^{(3)}$	$-.087155743 - .003805302\ i$
$A_0^{(3)}$	$.349048129 + .000000000\ i$	$A_0^{(3)}$	$.174311485 + .000000000\ i$
$A_1^{(3)}$	$-.174524064 + .001523048\ i$	$A_1^{(3)}$	$-.087155743 + .003805302\ i$
Four-Point		**Four-Point**	
$A_{-1}^{(4)}$	$.000000000 - .159468968\ i$	$A_{-1}^{(4)}$	$.000000000 - .198487531\ i$
$A_0^{(4)}$	$.008348504 + .478285473\ i$	$A_0^{(4)}$	$.051766326 + .591691817\ i$
$A_1^{(4)}$	$-.016694465 - .478066926\ i$	$A_1^{(4)}$	$-.103138680 - .584928518\ i$
$A_2^{(4)}$	$.008345961 + .159250422\ i$	$A_2^{(4)}$	$.051372353 + .191724233\ i$
Five-Point		**Five-Point**	
$A_{-2}^{(5)}$	$.033858948 + .001774472\ i$	$A_{-2}^{(5)}$	$.050675302 + .013578406\ i$
$A_{-1}^{(5)}$	$-.135549373 - .003549485\ i$	$A_{-1}^{(5)}$	$-.207067435 - .027260945\ i$
$A_0^{(5)}$	$.203380849 + .000000000\ i$	$A_0^{(5)}$	$.312784266 + .000000000\ i$
$A_1^{(5)}$	$-.135549373 + .003549485\ i$	$A_1^{(5)}$	$-.207067435 + .027260945\ i$
$A_2^{(5)}$	$.033858948 - .001774472\ i$	$A_2^{(5)}$	$.050675302 - .013578406\ i$
Six-Point		**Six-Point**	
$A_{-2}^{(6)}$	$-.001290203 + .029550516\ i$	$A_{-2}^{(6)}$	$-.004915368 + .022171794\ i$
$A_{-1}^{(6)}$	$.001290203 - .147842663\ i$	$A_{-1}^{(6)}$	$.004915368 - .112580444\ i$
$A_0^{(6)}$	$.007739255 + .295550237\ i$	$A_0^{(6)}$	$.029305450 + .222596993\ i$
$A_1^{(6)}$	$-.018049095 - .295100100\ i$	$A_1^{(6)}$	$-.067513797 - .214126406\ i$
$A_2^{(6)}$	$.014170631 + .147167629\ i$	$A_2^{(6)}$	$.052033388 + .099955208\ i$
$A_3^{(6)}$	$-.003860791 - .029325618\ i$	$A_3^{(6)}$	$-.013825042 - .018017145\ i$
Seven-Point		**Seven-Point**	
$A_{-3}^{(7)}$	$-.018698982 - .002461768\ i$	$A_{-3}^{(7)}$	$-.014168837 - .010872131\ i$
$A_{-2}^{(7)}$	$.112681322 + .009858338\ i$	$A_{-2}^{(7)}$	$.096041721 + .044784990\ i$
$A_{-1}^{(7)}$	$-.282434927 - .012331375\ i$	$A_{-1}^{(7)}$	$-.256925067 - .056958916\ i$
$A_0^{(7)}$	$.376905174 + .000000000\ i$	$A_0^{(7)}$	$.350104367 + .000000000\ i$
$A_1^{(7)}$	$-.282434927 + .012331375\ i$	$A_1^{(7)}$	$-.256925067 + .056958916\ i$
$A_2^{(7)}$	$.112681322 - .009858338\ i$	$A_2^{(7)}$	$.096041721 - .044784990\ i$
$A_3^{(7)}$	$-.018698982 + .002461768\ i$	$A_3^{(7)}$	$-.014168837 + .010872131\ i$
Eight-Point		**Eight-Point**	
$A_{-3}^{(8)}$	$.002806397 - .022856272\ i$	$A_{-3}^{(8)}$	$.006160708 - .008798403\ i$
$A_{-2}^{(8)}$	$-.011237575 + .160704814\ i$	$A_{-2}^{(8)}$	$-.025325223 + .069580479\ i$
$A_{-1}^{(8)}$	$.008431178 - .483021868\ i$	$A_{-1}^{(8)}$	$.019164515 - .219051406\ i$
$A_0^{(8)}$	$.028093940 + .804505186\ i$	$A_0^{(8)}$	$.063315659 + .359080946\ i$
$A_1^{(8)}$	$-.070159987 - .801932317\ i$	$A_1^{(8)}$	$-.154095218 - .330458261\ i$
$A_2^{(8)}$	$.067233891 + .478393994\ i$	$A_2^{(8)}$	$.141341376 + .168444093\ i$
$A_3^{(8)}$	$-.030738802 - .158137429\ i$	$A_3^{(8)}$	$-.060654932 - .042471040\ i$
$A_4^{(8)}$	$.005570958 + .022343891\ i$	$A_4^{(8)}$	$.010093114 + .003673593\ i$
Nine-Point		**Nine-Point**	
$A_{-4}^{(9)}$	$.004272099 + .001065154\ i$	$A_{-4}^{(9)}$	$.000856245 + .002352514\ i$
$A_{-3}^{(9)}$	$-.034605569 - .006413763\ i$	$A_{-3}^{(9)}$	$-.011949937 - .015573461\ i$
$A_{-2}^{(9)}$	$.122194187 + .015003560\ i$	$A_{-2}^{(9)}$	$.055479023 + .038846830\ i$
$A_{-1}^{(9)}$	$-.245680215 - .015026446\ i$	$A_{-1}^{(9)}$	$-.128080304 - .040383565\ i$
$A_0^{(9)}$	$.307638995 + .000000000\ i$	$A_0^{(9)}$	$.167389946 + .000000000\ i$
$A_1^{(9)}$	$-.245680215 + .015026446\ i$	$A_1^{(9)}$	$-.128080304 + .040383565\ i$
$A_2^{(9)}$	$.122194187 - .015003560\ i$	$A_2^{(9)}$	$.055479023 - .038846830\ i$
$A_3^{(9)}$	$-.034605569 + .006413763\ i$	$A_3^{(9)}$	$-.011949937 + .015573461\ i$
$A_4^{(9)}$	$.004272099 - .001065154\ i$	$A_4^{(9)}$	$.000856245 - .002352514\ i$

PART II—*Continued*

$\theta = 10°$		$\theta = 15°$	
Three-Point		**Three-Point**	
$A_{-1}^{(3)}$	$-.173648178 - .015192247\ i$	$A_{-1}^{(3)}$	$-.258819045 - .034074174\ i$
$A_0^{(3)}$	$.347296355 + .000000000\ i$	$A_0^{(3)}$	$.517638090 + .000000000\ i$
$A_1^{(3)}$	$-.173648178 + .015192247\ i$	$A_1^{(3)}$	$-.258819045 + .034074174\ i$
Four-Point		**Four-Point**	
$A_{-1}^{(4)}$	$.000000000 - .156683210\ i$	$A_{-1}^{(4)}$	$.000000000 - .051712264\ i$
$A_0^{(4)}$	$.080796568 + .458220106\ i$	$A_0^{(4)}$	$.039240251 + .146446609\ i$
$A_1^{(4)}$	$-.159138173 - .437228536\ i$	$A_1^{(4)}$	$-.075806343 - .131300438\ i$
$A_2^{(4)}$	$.078341605 + .135691640\ i$	$A_2^{(4)}$	$.036566092 + .036566092\ i$
Five-Point		**Five-Point**	
$A_{-2}^{(5)}$	$.028202446 + .016282690\ i$	$A_{-2}^{(5)}$	$.024919188 + .024919188\ i$
$A_{-1}^{(5)}$	$-.123439912 - .033075625\ i$	$A_{-1}^{(5)}$	$-.124720132 - .051660770\ i$
$A_0^{(5)}$	$.190474931 + .000000000\ i$	$A_0^{(5)}$	$.199601888 + .000000000\ i$
$A_1^{(5)}$	$-.123439912 + .033075625\ i$	$A_1^{(5)}$	$-.124720132 + .051660770\ i$
$A_2^{(5)}$	$.028202446 - .016282690\ i$	$A_2^{(5)}$	$.024919188 - .024919188\ i$
Six-Point		**Six-Point**	
$A_{-2}^{(6)}$	$-.011632757 + .024946528\ i$	$A_{-2}^{(6)}$	$-.026119998 + .034040243\ i$
$A_{-1}^{(6)}$	$.011632757 - .132963023\ i$	$A_{-1}^{(6)}$	$.026119998 - .198401078\ i$
$A_0^{(6)}$	$.068040005 + .253928757\ i$	$A_0^{(6)}$	$.147941118 + .357161454\ i$
$A_1^{(6)}$	$-.150785441 - .215343927\ i$	$A_1^{(6)}$	$-.306701494 - .235340333\ i$
$A_2^{(6)}$	$.109332977 + .076555775\ i$	$A_2^{(6)}$	$.198401078 + .026119998\ i$
$A_3^{(6)}$	$-.026587542 - .007124110\ i$	$A_3^{(6)}$	$-.039640703 + .016419717\ i$
Seven-Point		**Seven-Point**	
$A_{-3}^{(7)}$	$-.004296364 - .016034250\ i$	$A_{-3}^{(7)}$	$.003111022 - .007510672\ i$
$A_{-2}^{(7)}$	$.061213379 + .072951264\ i$	$A_{-2}^{(7)}$	$.011398482 + .042539714\ i$
$A_{-1}^{(7)}$	$-.210054793 - .097950159\ i$	$A_{-1}^{(7)}$	$-.082180418 - .063059252\ i$
$A_0^{(7)}$	$.306275557 + .000000000\ i$	$A_0^{(7)}$	$.135341826 + .000000000\ i$
$A_1^{(7)}$	$-.210054793 + .097950159\ i$	$A_1^{(7)}$	$-.082180418 + .063059252\ i$
$A_2^{(7)}$	$.061213379 - .072951264\ i$	$A_2^{(7)}$	$.011398482 - .042539714\ i$
$A_3^{(7)}$	$-.004296364 + .016034250\ i$	$A_3^{(7)}$	$.003111022 + .007510672\ i$
Eight-Point		**Eight-Point**	
$A_{-3}^{(8)}$	$.017894187 - .006512951\ i$	$A_{-3}^{(8)}$	$.012459594 + .003338538\ i$
$A_{-2}^{(8)}$	$-.080554340 + .096000925\ i$	$A_{-2}^{(8)}$	$-.067898429 + .039201176\ i$
$A_{-1}^{(8)}$	$.062660153 - .355363389\ i$	$A_{-1}^{(8)}$	$.055438835 - .206900549\ i$
$A_0^{(8)}$	$.201523147 + .553680297\ i$	$A_0^{(8)}$	$.170370869 + .295091000\ i$
$A_1^{(8)}$	$-.451364313 - .378739629\ i$	$A_1^{(8)}$	$-.329131244 - .088190451\ i$
$A_2^{(8)}$	$.355363389 + .062660153\ i$	$A_2^{(8)}$	$.185501958 - .107099605\ i$
$A_3^{(8)}$	$-.117762567 + .042862069\ i$	$A_3^{(8)}$	$-.020292022 + .075730857\ i$
$A_4^{(8)}$	$.012240345 - .014587475\ i$	$A_4^{(8)}$	$-.006449560 - .011170966\ i$
Nine-Point		**Nine-Point**	
$A_{-4}^{(9)}$	$-.002512465 + .002108209\ i$	$A_{-4}^{(9)}$	$-.005667097 - .003271900\ i$
$A_{-3}^{(9)}$	$.006260569 - .023364763\ i$	$A_{-3}^{(9)}$	$.040112374 - .016615089\ i$
$A_{-2}^{(9)}$	$.027326899 + .075080038\ i$	$A_{-2}^{(9)}$	$-.034445276 + .128551520\ i$
$A_{-1}^{(9)}$	$-.126437808 - .088532706\ i$	$A_{-1}^{(9)}$	$-.149701416 - .195094684\ i$
$A_0^{(9)}$	$.190725610 + .000000000\ i$	$A_0^{(9)}$	$.299402832 + .000000000\ i$
$A_1^{(9)}$	$-.126437808 + .088532706\ i$	$A_1^{(9)}$	$-.149701416 + .195094684\ i$
$A_2^{(9)}$	$.027326899 - .075080038\ i$	$A_2^{(9)}$	$-.034445276 - .128551520\ i$
$A_3^{(9)}$	$.006260569 + .023364763\ i$	$A_3^{(9)}$	$.040112374 + .016615089\ i$
$A_4^{(9)}$	$-.002512465 - .002108209\ i$	$A_4^{(9)}$	$-.005667097 + .003271900\ i$

PART II—*Concluded*

	$\theta = 20°$		$\theta = 30°$	
	Three-Point		Three-Point	
$A_{-1}^{(3)}$	−.342020143	−.060307379 i	−.050000000	−.013397460 i
$A_0^{(3)}$	.684040287	+.000000000 i	.100000000	+.000000000 i
$A_1^{(3)}$	−.342020143	+.060307379 i	−.050000000	+.013397460 i
	Four-Point		Four-Point	
$A_{-1}^{(4)}$	.000000000	−.118782349 i	.000000000	−.366025404 i
$A_0^{(4)}$	.116977778	+.321393805 i	.500000000	+.866025404 i
$A_1^{(4)}$	−.219846310	−.262002630 i	−.866025404	−.500000000 i
$A_2^{(4)}$	.102868532	+.059391175 i	.366025404	+.000000000 i
	Five-Point		Five-Point	
$A_{-2}^{(5)}$	.092091566	+.159507271 i	.000000000	+.169872981 i
$A_{-1}^{(5)}$	−.590442693	−.340892248 i	−.401923789	−.401923789 i
$A_0^{(5)}$	.996702254	+.000000000 i	.803847577	+.000000000 i
$A_1^{(5)}$	−.590442693	+.340892248 i	−.401923789	+.401923789 i
$A_2^{(5)}$	.092091566	−.159507271 i	.000000000	−.169872981 i
	Six-Point		Six-Point	
$A_{-2}^{(6)}$	−.021616620	+.018138498 i	−.031698730	+.008493649 i
$A_{-1}^{(6)}$	.021616620	−.122593943 i	.031698730	−.118301270 i
$A_0^{(6)}$	.116977778	+.202611455 i	.150000000	+.150000000 i
$A_1^{(6)}$	−.219846310	−.080017513 i	−.204903811	+.054903811 i
$A_2^{(6)}$	.116977778	−.042576429 i	.031698730	−.118301270 i
$A_3^{(6)}$	−.014109246	+.024437932 i	.023205081	+.023205081 i
	Seven-Point		Seven-Point	
$A_{-3}^{(7)}$	.009902810	−.005717390 i	.023205081	+.023205081 i
$A_{-2}^{(7)}$	−.009902810	+.056161626 i	−.109807621	+.063397460 i
$A_{-1}^{(7)}$	−.082102804	−.097846311 i	−.063397460	−.236602540 i
$A_0^{(7)}$	.164205608	+.000000000 i	.300000000	+.000000000 i
$A_1^{(7)}$	−.082102804	+.097846311 i	−.063397460	+.236602540 i
$A_2^{(7)}$	−.009902810	−.056161626 i	−.109807621	−.063397460 i
$A_3^{(7)}$	.009902810	+.005717390 i	.023205081	−.023205081 i
	Eight-Point		Eight-Point	
$A_{-3}^{(8)}$	.013813735	+.016462568 i	−.031698730	+.054903811 i
$A_{-2}^{(8)}$	−.114527733	+.020194329 i	−.204903811	−.118301270 i
$A_{-1}^{(8)}$	.100713999	−.276709437 i	.236602540	−.409807621 i
$A_0^{(8)}$	.289994401	+.345601870 i	.559807621	+.323205081 i
$A_1^{(8)}$	−.444297199	+.078341584 i	−.323205081	+.559807621 i
$A_2^{(8)}$	.100713999	−.276709437 i	−.409807621	−.236602540 i
$A_3^{(8)}$	.074752669	+.089086762 i	.118301270	−.204903811 i
$A_4^{(8)}$	−.021163870	+.003731761 i	.054903811	+.031698730 i
	Nine-Point		Nine-Point	
$A_{-4}^{(9)}$	.003675067	−.020842343 i	.005490381	+.009509619 i
$A_{-3}^{(9)}$	.103945799	+.060013135 i	−.025980762	+.025980762 i
$A_{-2}^{(9)}$	−.252618067	+.211971726 i	−.061471143	−.035490381 i
$A_{-1}^{(9)}$	−.195354200	−.536731253 i	.025980762	−.096961524 i
$A_0^{(9)}$	.680702801	+.000000000 i	.111961524	+.000000000 i
$A_1^{(9)}$	−.195354200	+.536731253 i	.025980762	+.096961524 i
$A_2^{(9)}$	−.252618067	−.211971726 i	−.061471143	+.035490381 i
$A_3^{(9)}$	.103945799	−.060013135 i	−.025980762	−.025980762 i
$A_4^{(9)}$	.003675067	+.020842343 i	.005490381	−.009509619 i

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ON THE TRANSONIC FLOW OF A COMPRESSIBLE FLUID THROUGH AN AXIALLY SYMMETRICAL NOZZLE

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1. Introduction and summary. The transonic flow through a converging and diverging nozzle with circular cross-section was studied experimentally by Stanton and theoretically by Hooker and Görtler. According to Stanton's observations [1], when the inlet pressure (i.e., the initial pressure in drums) is so low that the speed of sound is not attained anywhere in the flow, the fluid velocity diminishes on passing the throat of a nozzle and the axial velocity distribution is approximately symmetrical on either side of the throat, but, when the inlet pressure is increased, the velocity of sound is ultimately attained by the whole field of flow after passing the throat and the axial velocity distribution becomes asymmetrical about it.

In connection with Stanton's experiments, Hooker [2] developed a theoretical analysis of the flow in the neighbourhood of the throat of a constriction in a circular wind channel, by using the method of treatment developed by Taylor [3] for the two-dimensional case, and it has thus been found that there appears to be a limit to the velocity at the centre of the constriction in order that the flow symmetrical about the axis of the nozzle as well as its throat can exist and also that the asymmetrical type of flow is unique. Comparing his theoretical results with Stanton's observations, Hooker has found good agreements between them.

Later, a similar theoretical analysis was developed by Görtler [4], who, by employing a method which is essentially the same as that used by Hooker, has obtained the results which are capable of explaining qualitatively the transient phenomena of the flows in a given Laval nozzle as observed by Stanton. However, it seems rather difficult to draw any definite conclusion from the results of his analysis, since he has used only approximate solutions in the form of series in the coordinates, which become rapidly divergent when the transition state is approached. Moreover, it seems that there are unfortunately some ambiguities in his analysis.

So far, no definite theory seems to have been developed yet which has dealt with the question as to the manner in which the limiting form of the symmetrical type of flow is to be related to the unique asymmetrical one. It is therefore desirable to solve theoretically the transient phenomena from the one type to the other as a problem of non-stationary flow, but it is at present hopeless to do so owing to the great mathematical difficulties involved.

Recently, Tamada and one of the present writers [5] have developed a theory of the two-dimensional steady transonic flows of compressible fluid, by introducing a new hypothetical gas which can approximate the real gas, subject to the adiabatic law, in the vicinity of the critical state where the fluid velocity becomes just equal to the local velocity of sound. The flow of such a hypothetical gas is governed, in an exact manner, by a rather simple fundamental non-linear equation of motion of the mixed type which is essentially equivalent to a simplified

fundamental equation of motion deduced by von Kármán [6] in his recent paper on the similarity law of transonic flow. It has been found that the equation can be solved exactly in certain cases, and by making use of one of the exact particular solutions so obtained, we have discussed the transonic flow in a two-dimensional converging and diverging nozzle and thus explained theoretically, on the standpoint of view of "quasi-stationary" flows, the possible transition from the approximately symmetrical type of flow (i.e., flow of Taylor's type) to the asymmetrical one (i.e., flow of Meyer's type).

In the present paper, a similar discussion is made on the steady irrotational transonic flow of the real gas, subject to the adiabatic law, through an axially symmetrical nozzle with circular cross-section. To do this, use is made of a simplified fundamental equation of motion for an axially symmetrical flow which also has been deduced by von Kármán in the paper above referred to and the said equation is solved by employing a method analogous to the one used by us in the previous paper.

By carrying out detailed numerical calculations, it is shown that in accordance with the results of Stanton's observations, there appears a limit to the occurrence of approximately symmetrical type of flow, and further, we can explain satisfactorily the manner in which the limiting form of the symmetrical type of flow can be continued to the asymmetrical unique flow.

2. The fundamental equation. The differential equation for the velocity potential Φ in axially symmetrical irrotational flow of a compressible fluid obeying the adiabatic law is given by

$$\left(1-\frac{u^2}{c^2}\right)\Phi_{xx}-\frac{2uv}{c^2}\Phi_{xy}+\frac{1}{y}\Phi_y+\left(1-\frac{v^2}{c^2}\right)\Phi_{yy}=0, \tag{2.1}$$

where x is the abscissa taken along the axis of symmetry and y the radial distance from the axis, and u, v are the velocity components in the x and y directions so that

$$u=\Phi_x, \qquad v=\Phi_y. \tag{2.2}$$

Further, c is the velocity of sound at the speed $(u^2+v^2)^{1/2}$, given by

$$c^2=\frac{\gamma+1}{2}c_*^2-\frac{\gamma-1}{2}(u^2+v^2), \tag{2.3}$$

where c_* is the so-called critical sound velocity, that is, the speed of sound at points where the local Mach number is unity, and γ is the ratio of two specific heats of the fluid.

Now, confining ourselves to the transonic region, we consider the case where all flow velocities differ only slightly from the critical velocity of sound and we assume that

$$u=c_*+\phi_x, \qquad v=\phi_y, \tag{2.4}$$

where ϕ_x and ϕ_y are the small perturbation velocities derived from the perturbation velocity potential ϕ.

Introducing (2.2), (2.3) and (2.4) into (2.1) and neglecting terms of higher order than the second in the derivatives of ϕ, we get the following simplified differential equation:

$$\phi_{yy} + \frac{1}{y}\phi_y = \frac{\gamma+1}{c_*}\phi_x\phi_{xx} + \frac{2}{c_*}\phi_y\phi_{xy}. \tag{2.5}$$

For simplicity we put

$$\phi = c_* f, \qquad x = \xi, \qquad y = \sqrt{2/(\gamma+1)}\,\eta. \tag{2.6}$$

Then, Eq. (2.5) transforms into

$$f_{\eta\eta} + \eta^{-1} f_\eta = (f_\xi^2 + f_\eta^2)_\xi. \tag{2.7}$$

We now assume that the second term $(f_\eta^2)_\xi$ on the right-hand side can be neglected in comparison with the first term $(f_\xi^2)_\xi$. This simplification may be considered to be justifiable in the case of a nozzle with slowly varying cross-section along its axis, which will be discussed presently, since in such a case $|(f_\xi^2)_\xi| >> |(f_\eta^2)_\xi|$ in the greater part of the field of flow. Moreover, it is to be noted that this approximation causes no change in the type of the partial differential equation so far as $|2f_\xi| > f_\eta^2$, which is the case in general, because the type of Eq. (2.7) is characterized by the sign of the expression $2f_\xi + f_\eta^2$, while that of the simplified equation is determined by the sign of f_ξ. Making such an approximation we have ultimately

$$f_{\eta\eta} + \eta^{-1} f_\eta = 2f_\xi f_{\xi\xi}. \tag{2.8}$$

This equation is exactly the same as that of von Kármán who, for the purpose of obtaining the similarity law of transonic flow past slender bodies, has deduced this by introducing an affine transformation other than (2.6) and by taking the appropriate boundary conditions into account.

Differentiating (2.8) with respect to ξ and putting

$$f_\xi = w, \tag{2.9}$$

we get

$$w_{\eta\eta} + \eta^{-1} w_\eta = (w^2)_{\xi\xi}, \tag{2.10}$$

and this is the basic equation for our analysis in the following lines.

Here, it may be noted that the above equation (2.10) degenerates into the basic equation as adopted by Tamada and one of the present writers, in the previous paper already referred to, in the discussions of two-dimensional transonic flows of a hypothetical gas, if the second term on the left-hand side be omitted and w, ξ, η be replaced by kw, φ, ψ respectively, where ψ denotes the stream function.

3. Some exact solutions of the fundamental equation (2.10). The fundamental equation (2.10) can be solved exactly by assuming w in the following forms:

(a) $w = X(\xi) + Y(\eta)$,

(b) $w = X(\xi)Y(\eta)$,

(c) $w = Y_0(\eta) + Y_1(\eta)\xi^2$,

(d) $w = Z(z) + 2\mu^2\eta^2, \qquad z = \xi + \mu\eta^2$,

where μ is a certain constant.

These solutions together with the corresponding flow patterns have been investigated in detail, but only a brief sketch of the results obtained will be given here for the first three cases. The remaining case will be discussed in some detail in the subsequent sections.

The first exact solution (a) gives, as a particular case, the flow through a converging and diverging nozzle with circular cross-section, which changes from the subsonic state to the supersonic one as it passes through the throat of the nozzle. As will be seen later, however, this case is nothing but a special case of the flows represented by the fourth solution (d).

The second solution (b) also gives, as an example, the flow through a straight circular nozzle with one constriction, while the third solution (c) includes the flow through a straight circular nozzle with two contractions.

4. Flow through a constriction in a nozzle with circular cross-section. As the last example, we shall consider the flow represented by the fourth solution (d) of the form:

$$w = Z(z) + 2\mu^2\eta^2, \tag{4.1}$$

where

$$z = \xi + \mu\eta^2, \tag{4.2}$$

and μ is a certain constant.

Inserting this expression for w into the basic equation (2.10), we obtain the following equation for determining the function $Z(z)$:

$$\frac{d}{dz}\left(Z\frac{dZ}{dz}\right) - 2\mu\frac{dZ}{dz} - 4\mu^2 = 0. \tag{4.3}$$

This equation can be integrated once immediately and we get

$$Z(dZ/dz) - 2\mu Z - 4\mu^2 z = 0, \tag{4.4}$$

where a constant of integration has been included in z for the sake of convenience.

Integrating Eq. (4.4) again we have ultimately

$$\left.\begin{aligned}(Z - \mu\omega_1 z)^{\omega_1}(Z - \mu\omega_2 z)^{-\omega_2} &= a,\\ \omega_1 = 1 + \sqrt{5}, \qquad \omega_2 &= 1 - \sqrt{5},\end{aligned}\right\} \tag{4.5}$$

where a is an arbitrary constant, real or complex.

It will readily be seen that this solution includes the preceding first solution (a) as a special case when $a = 0$.

The velocity components u, v in this case are obtained by (2.4), (2.6), (2.9)

and (4.1) in the forms:

$$\left.\begin{aligned} u &= c_*(1 + f_\xi), \qquad v = c_*\sqrt{(\gamma+1)/2}\, f_\eta, \\ \text{with} \quad f_\xi &= 2\mu^2\eta^2 + Z, \qquad f_\eta = 2\mu\eta\{\mu(2\xi + \mu\eta^2) + Z\}, \end{aligned}\right\} \tag{4.6}$$

and from the expression for u it is readily seen that the value of Z gives the rate of excess of the axial velocity over the critical speed of sound c_*, since f_ξ is equal to Z on the axis ($\eta = 0$) of a nozzle.

Before proceeding further, we shall now discuss the properties of the function $Z(z)$ in relation to the types of flow in a nozzle.

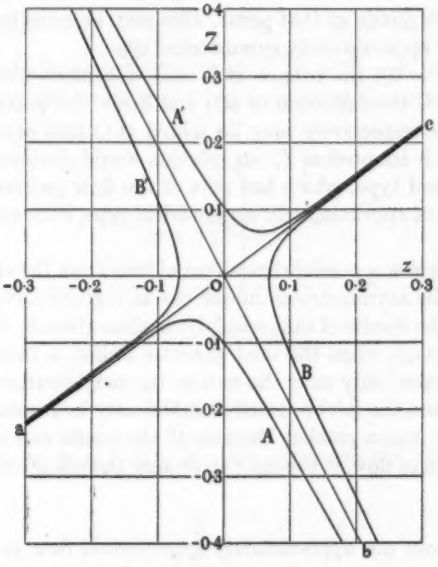

FIG. 1.

The two arbitrary constants μ and a involved in the function $Z(z)$ should be determined by appropriate boundary conditions which will be considered presently. For certain prescribed values of μ and a, Eq. (4.5) gives us curves of Z plotted against z as shown in Fig. 1, where we have assumed $\mu = -0.6$ and $|a| = (0.1)^{\omega_1-\omega_2} = (0.1)^{2\sqrt{5}}$. The four branches A, B, A' and B' correspond to $a = |a|e^{i\pi(\omega_1-\omega_2)}$, $a = |a|e^{-i\pi\omega_2}$, $a = |a|$ and $a = |a|e^{i\pi\omega_1}$ respectively, and the two asymptotes are given by $Z = \mu\omega_1 z$ and $Z = \mu\omega_2 z$.

First, if the branches B and B' are adopted as Z, there appears in each case a shock line in the meridian plane, since dZ/dz becomes infinite at points where $Z = 0$. The discussions of the flows in these cases will be omitted here, however, since they are of less practical interest.

Next, if the branch A' is adopted as Z, then Z is always positive and therefore

the whole field of flow is supersonic. If, on the other hand, the branch A is adopted, Z is always negative and the flow is subsonic on either side of the throat of the nozzle but may contain a limited supersonic region in the neighbourhood of the wall at the throat. In this case, the axial velocity distribution is approximately symmetrical on either side of the throat and has its maximum in the neighbourhood of the throat.

When the value of the constant a is varied, the branch A gradually changes its form and it coincides with the two asymptotes aO and bO as the value of a tends to zero. In the limiting case when $a = 0$, the local speed of sound is attained at some point (which corresponds to the point O in Fig. 1) on the axis of the nozzle; hence the vertex of the limited supersonic region becomes in contact with the axis of the nozzle at that point. This may be considered as the limiting case of the flow of approximately symmetrical type.

In this case ($a = 0$), the branch B is also coincident with the asymptotes, bO and cO (which is the extension of aO) and hence the parts aO and Oc of the branches A and B respectively may be joined at O into one straight line aOc. If this branch aOc is adopted as Z, our solution would obviously give a flow pattern of asymmetrical type, which has part of the flow pattern in common with the preceding flow of approximately symmetrical type, corresponding to the half-branch aO of Z.

These results suggest a possible transition of flow from the approximately symmetrical type to the asymmetrical unique one in a given Laval nozzle. Thus, as will be seen from the results of numerical discussions given in the subsequent section, at the first stage when the inlet pressure is low, a limited supersonic region, if any, is located only near the wall in the neighbourhood of the throat of the nozzle, but, when the inlet pressure is sufficiently increased, the vertex of the limited supersonic region reaches the axis of the nozzle and subsequently, after the change of types of flow, it becomes to occupy the whole field downstream the throat.

5. Transition from the approximately symmetrical flow to the asymmetrical one in a given circular nozzle. Stanton has made his experiments on the flow of air through a constriction in a circular wind channel, which consists of a symmetrical circular sleeve whose longitudinal section is two circular arcs of radius R of minimum distance $2h$ apart. The numerical value of the ratio h/R in Stanton's channel was 0.2100.

This particular value of $h/R = 0.2100$ will be taken in the subsequent analysis in order that a direct comparison with Stanton's results might be possible. Also, we shall take, with Stanton, the value of 1.408 for the ratio of two specific heats of the air.

Now, denoting by (x_0, y_0) the point on the wall at the throat of a nozzle (cf. Fig. 2) where y_0 is equal to h, we assume that the longitudinal section of the wall of the nozzle satisfies the conditions:

$$\frac{dy}{dx} = 0, \qquad \frac{d^2y}{dx^2} = \frac{0.2100}{h} \quad \text{at} \quad x = x_0, \qquad y = y_0. \tag{5.1}$$

The equation for the streamlines is given by

$$\frac{dy}{dx} = \frac{v}{u} = \sqrt{\frac{\gamma+1}{2}}\,\frac{2\mu\eta\{\mu(z+\xi)+Z\}}{1+2\mu^2\eta^2+Z}, \tag{5.2}$$

and if use is made of this equation, the conditions (5.1) can be written as:

$$\left.\begin{aligned} &\mu(z+\xi)+Z=0,\\ &8\mu^2\eta\sqrt{(\gamma+1)/2}\,(1+\mu z/Z)=(.21/h)(1+2\mu^2\eta^2+Z),\\ &\text{at}\quad x=x_0,\qquad y=y_0. \end{aligned}\right\} \tag{5.3}$$

For convenience' sake, we now put

$$2\mu z=-\sigma z',\qquad 2\mu\xi=-\sigma\xi',\qquad Z=-\sigma Z', \tag{5.4}$$

where $\sigma=(Z)_{z=0}$ which is equivalent to $\sigma^{\omega_1-\omega_2}=\sigma^{2\sqrt{5}}=a$, and this σ gives the rate of excess of the axial velocity over the critical speed of sound c_* at the origin of the x, y plane.

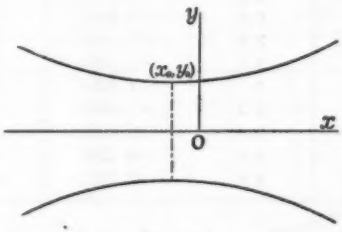

FIG. 2.

Then, putting $y_0=h=1$ for the sake of simplicity, the conditions (5.3) transform into

$$\left.\begin{aligned} &\mu^2=-\frac{2}{\gamma+1}\,\sigma(Z_0'+z_0'),\\ &\frac{1}{\sigma}=3Z_0'+2z_0'-\frac{4}{0.2100}\,\frac{(Z_0'+z_0')(2Z_0'+z_0')}{Z_0'}, \end{aligned}\right\} \tag{5.5}$$

where z_0' and Z_0' are the values of z' and Z' at the point $x=x_0$, $y=y_0$ $(=1)$ respectively. Also, with the new variables z' and Z', Eq. (4.5) reduces to

$$\left(-Z'+\frac{\omega_1}{2}z'\right)^{\omega_1}\left(-Z'+\frac{\omega_2}{2}z'\right)^{-\omega_2}=1. \tag{5.6}$$

With the aid of this equation the values of Z' have been calculated for various values of z'. Some of them are given in Table I, and the curves of Z' plotted against z' are shown in Fig. 3.

If a suitable value is given to z_0', the corresponding value of Z_0' can be determined from Table I or Fig. 3, and then the values of σ and μ can be obtained with the aid of (5.5). Table II shows the values of σ and μ thus determined, by taking

$\gamma = 1.408$ for air, in three cases in which z_0' is equal to 1.5, 2 and 4 respectively. In this table, the values of ξ_0' and ξ_0 $(=x_0)$ are also shown, which have been calculated by the relations:

$$\xi_0' = -(2Z_0' + z_0'), \qquad \xi_0(=x_0) = -\frac{\sigma}{2\mu}\,\xi_0'. \tag{5.7}$$

TABLE I

Values of Z' against z'

z'	Z'	z'	Z'	z'	Z'
−2	−3.7763	1	−0.7256	4	−2.4754
−1.8	−3.4715	1.2	−0.8122	4.4	−2.7219
−1.6	−3.1691	1.4	−0.9137	4.8	−2.9686
−1.4	−2.8698	1.6	−1.0235	5.2	−3.2154
−1.2	−2.5743	1.8	−1.1381	5.6	−3.4623
−1	−2.2837	2	−1.2557	6	−3.7093
−0.8	−1.9998	2.2	−1.3750	6.4	−3.9564
−0.6	−1.7247	2.4	−1.4955	6.8	−4.2034
−0.4	−1.4621	2.6	−1.6168	7.2	−4.4505
−0.2	−1.2175	2.8	−1.7387	7.6	−4.6977
0	−1	3	−1.8609	8	−4.9448
0.2	−0.8228	3.2	−1.9835	8.5	−5.2537
0.4	−0.7029	3.4	−2.1063	9	−5.5627
0.6	−0.6523	3.6	−2.2292	9.5	−5.8717
0.8	−0.6660	3.8	−2.3522	10	−6.1806

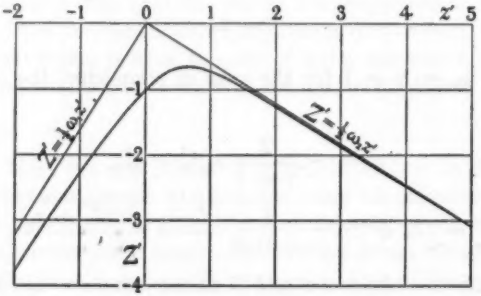

FIG. 3.

The shape of the longitudinal section of the boundary wall of the nozzle, which coincides with that in Stanton's channel as far as its curvature at the minimum constriction, can be found by integrating Eq. (5.2) subject to the conditions (5.1).

Figure 4 shows the shapes of the boundary wall as calculated for three cases in which the value of z_0' is equal to 1.5, 2 and 4 respectively, by taking $x - x_0$ and y as abscissa and ordinate respectively, the ordinate being shown on an enlarged scale. In this figure, the longitudinal section of the constriction in Stan-

ton's channel is also shown by a dotted-line curve for the sake of comparison, which represents the circle defined by $(x - x_0)^2 + (y - 1 - R)^2 = R^2$ where $R = 1/0.2100$. From this figure it will be seen that there is very satisfactory coincidence between these boundary curves within the range $|x - x_0| < 0.4$, especially in the upstream subsonic region.

The axial velocity distributions in the three cases in which z_0' is equal to 1.5, 2 and 4 respectively are shown in Fig. 5. It will be seen that in these cases the flow is approximately symmetrical about the throat of the nozzle.

The longitudinal section of the so-called sonic surface on which the fluid velocity is just equal to the local speed of sound, can be obtained approximately

TABLE II

z_0'	σ	μ	ξ_0'	$\xi_0(=x_0)$
1.5	−0.2239	−0.3147	0.4356	−0.1550
2	−0.1805	−0.3341	0.5113	−0.1382
4	−0.0954	−0.3459	0.9507	−0.1299
∞	0	−0.3471	∞	−0.1291

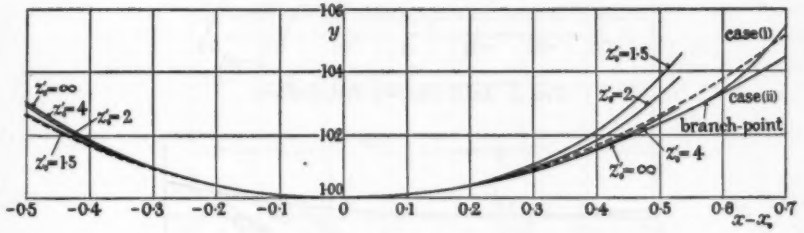

FIG. 4.

by putting $w = 0$, that is, by putting $Z + 2\mu^2\eta^2 = 0$. Thus, the equation for the longitudinal section of the sonic surface is given by

$$x = -(\sigma/2\mu)(z' + Z'), \qquad y = -\mu^{-1}\sqrt{(\sigma/[\gamma + 1])Z'}. \tag{5.8}$$

Figure 6 shows the longitudinal section of the sonic surface for the cases when $z_0' = 1.5, 2, 4$.

So far we have dealt with such flows that are approximately symmetrical about the minimum constriction of the nozzle. As will be seen from Figs. 5 and 6, these flows approach a limiting flow as the value of the parameter z_0' increases, that is, as the axial velocity at the throat increases. This limiting case of the approximately symmetrical flow is expressed by the solutions:

$$Z = \mu\omega_2 z \ (z < 0), \qquad Z = \mu\omega_1 z \ (z > 0), \tag{i}$$

which correspond to the asymptotes aO and bO in Fig. 1 respectively.

On the other hand, we have the following solution:

$$Z = \mu\omega_2 z \qquad \text{for all values of } z, \tag{ii}$$

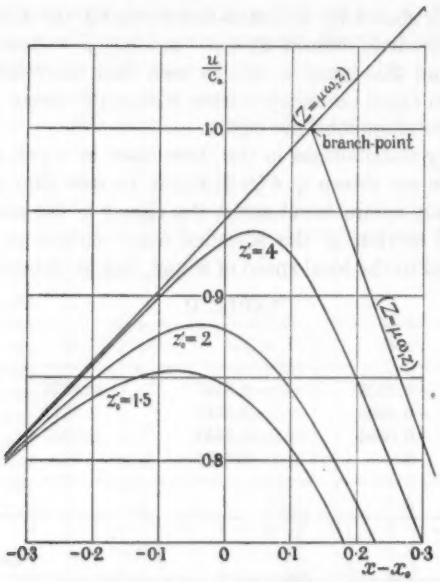

FIG. 5. Axial velocity distributions

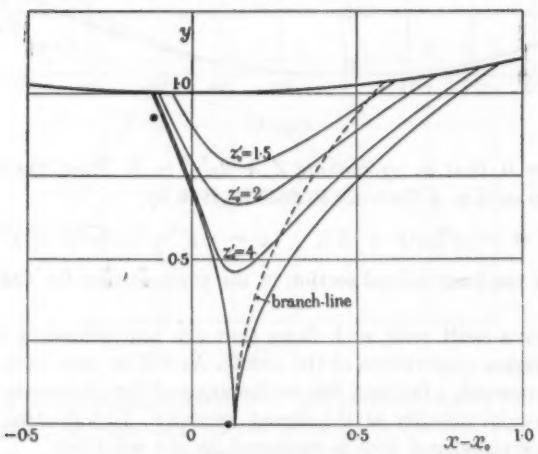

FIG. 6. Sonic lines

which corresponds to the asymptote *aOc* in Fig. 1 and gives the completely asymmetrical flow. It can easily be verified that these solutions (i) and (ii) are exactly of the same form as the first exact solution (a) mentioned in §3.

The branch $Z = \mu\omega_1 z$ $(z > 0)$ in case (i) and the branch $Z = \mu\omega_2 z$ $(z > 0)$ in case (ii) are both connected to the branch $Z = \mu\omega_2 z$ $(z < 0)$ at the origin of the z, Z plane (Fig. 1). In the longitudinal section of the field of flow, the points where $z = Z = 0$ form themselves into a curve which we shall call the "branch-line." At any point on such a branch-line, the velocity vectors are continuous in both cases (i) and (ii), but the accelerations in case (i) are discontinuous there. Corresponding to these properties it is found that the tangents to these streamlines in both cases (i) and (ii) are continuous at any point on the branch-line, and that the curvatures in case (ii) are continuous there, whilst those in case (i) are discontinuous except at the point of intersection of the branch-line with the axis.

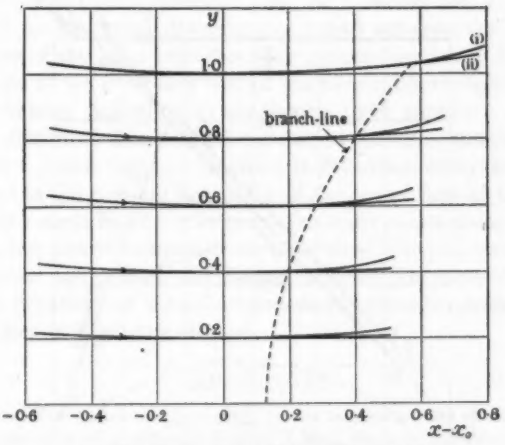

FIG. 7. Streamlines in cases (i) and (ii)

The values of σ, μ, ξ_0' and $\xi_0 (= x_0)$ for the limiting case $z_0' = \infty$ can be determined with the aid of (5.5) and (5.7), by considering the limit $z_0' \to \infty$. The results are shown in the last row of Table II.

Using the value of μ thus determined, the streamlines in the x, y plane have been obtained by integrating Eq. (5.2) in both cases (i) and (ii). The results are shown in Fig. 7, where the dotted-line curve represents the branch-line:

$$z = 0, \quad \text{i.e.,} \quad x = -\tfrac{1}{2}(\gamma + 1)\mu y^2.$$

After passing over this branch-line, the streamlines in case (i) depart from the axis of the nozzle much more than those in case (ii).

The axial velocity distribution in case (i) is expressed by the formulas:

$$u = c_*(1 + \mu\omega_2 x) \qquad (x < 0),$$

$$u = c_*(1 + \mu\omega_1 x) \qquad (x > 0),$$

while in case (ii) it is represented by

$$u = c_s(1 + \mu\omega_3 x) \qquad \text{for all values of } x.$$

These distributions are shown in the preceding Fig. 5.

6. Comparison with Stanton's observations. As mentioned already, Stanton carried out his observations on the flow through a constriction in a wind channel which consists of a symmetrical circular sleeve whose longitudinal section is two circular arcs of radius 1.385/0.2100 inches of minimum distance 2×1.385 inches apart. In the foregoing analysis, we have determined the shape of the

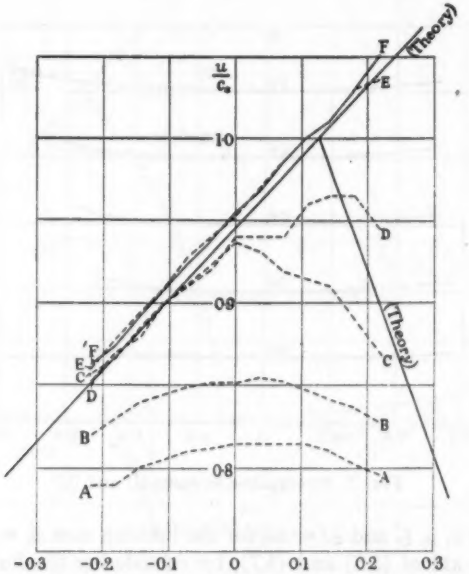

FIG. 8. Axial velocity distributions observed by Stanton

boundary wall of a nozzle in such a manner that the curvature of its longitudinal section at the minimum constriction coincides with that in Stanton's channel. As shown in Fig. 3, the calculated boundary curves of the nozzle differ only slightly from Stanton's circular boundary, especially in the upstream side of the throat, except for cases in which the value of the parameter z_0' is fairly small. Therefore, the results of our theoretical analysis can be directly compared with the results of Stanton's observations.

The observed axial velocity distributions are reproduced in Fig. 8, where the theoretical axial velocity distributions for the limiting case of approximately symmetrical flow as well as that for the asymmetrical flow are also shown for comparison. From this figure it is found that when the velocity at the centre

of the throat, $u_{x=x_0,y=0}$, is small, the observed axial velocity distributions are approximately symmetrical about the throat of the nozzle, and as the value of $u_{x=x_0,y=0}$ increases, the approximately symmetrical distribution curves are gradually shifted towards the downstream direction, the value of $u_{x=x_0,y=0}$ tending at the same time to a limiting value. Also, it is found that at the final stage, the flow pattern becomes completely asymmetrical and the axial velocity distribution increases almost linearly in the downstream direction of the axis.

Comparing these results of Stanton's observations with our theoretical results shown in Fig. 5, it will be seen that there is a good qualitative agreement between them, especially in the case of completely asymmetrical flow (case *F* in Stanton's experiments) and that the slope of the theoretical axial velocity distribution curve is in good agreement with the observed one over a wide range of values of $x - x_0$. Small discrepancies found between the theoretical and observed values of the axial velocities may perhaps be due to the fact that the calculated shape of the boundary wall of the nozzle differs slightly from the one in Stanton's channel. According to our theory, there appears a peaked curve of axial velocity distribution in the limiting case of the approximately symmetrical flow, but such a curve was not obtained in Stanton's observations. This may perhaps be due to the possible instability of the actual flow at this stage.

In Fig. 6, two small black circles show the sonic points found experimentally by Stanton in the case of completely asymmetrical flow (i.e., case *F*). The slight deviation of these points from the theoretical sonic line seems to be due to the effect of slight departure of the calculated shape of the boundary wall from the circular boundary in Stanton's channel.

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BEST APPROXIMATE INTEGRATION FORMULAS

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1. Introduction. Let $A = c_0x(0) + c_1x(1) + \cdots + c_mx(m)$ be an approximation of $\int_0^m x(t)\,dt$ which is exact whenever $x = x(t)$ is a polynomial in t of degree n, where m and n are integers, $m \geqq 1$, $n \geqq 0$, and $c_0, \cdots, c_m$ are constants; let the class of all such approximations A be denoted by $\mathfrak{A}_{m,n}$. For each $A \in \mathfrak{A}_{m,n}$, there is a kernel function $k(t)$ such that

$$R[x] = \int_0^m x(t)\,dt - A = \int_0^m x^{(n+1)}(t)k(t)\,dt$$

whenever the function x has a continuous $(n+1)$th derivative $x^{(n+1)}$. Indeed

$$k(t') = R[\psi_{t'}] = -R[\varphi_{t'}], \tag{1}$$

where

$$\psi_{t'} = \psi_{t'}(t) = \begin{cases} 0 \\ (t-t')^n/n! \end{cases}, \qquad \varphi_{t'} = \varphi_{t'}(t) = \begin{cases} (t-t')^n/n! & \text{if } t \leqq t', \\ 0 & \text{if } t > t'. \end{cases}$$

Schwarz's inequality thus affords the appraisal:

$$|R[x]| \leqq [J\int_0^m x^{(n+1)}(t)^2\,dt]^{\frac{1}{2}},$$

where

$$J = \int_0^m k^2(t)\,dt.$$

We say that that approximation A in $\mathfrak{A}_{m,n}$ is *best* which minimizes J. Best formulas were introduced by one of us in the paper [1]. The reader will find in [1] a discussion of the theory of best formulas, a discussion of the philosophy of their use, an indication of the method of their determination, and the best approximate integration formulas themselves for the cases $n = 0$, all m; and $n = 1, 2, 3$, $m \leqq 6$.

Here we present (§2) the best integration formulas for $n = 1$, $m \leqq 20$; $n = 2$, $m \leqq 12$; and $n = 3$, $n \leqq 9$; together with the corresponding values of J. Since the formulas are symmetrical: $c_i = c_{m-i}$, only those c_i are tabulated for which $2i \leqq m$. Although the derivation of these formulas involves an extended calculation, their use is direct and free of complexity.

The fundamental algebraic relationships are given in §3.

For the case $n = 1$, certain recursive relations (§4) afford a complete description of the best integration formulas for all m.

Finally in §5 we state certain unsolved problems.

2. The best formulas. As an illustration of the use of the following table, the best formula in $\mathfrak{A}_{3,1}$ is

$$A = 0.4[x(0) + x(3)] + 1.1[x(1) + x(2)];$$

[1] The authors gratefully acknowledge financial support received from the Office of Naval Research under contract with Queens College.

for this formula

$$|R[x]| \leqq \left[\int_0^m x^{(2)}(t)^2\, dt/120\right]^{\frac{1}{2}},$$

for any function x with continuous second derivative.

Best Approximate Integration Formulas

$n = 1.$

m	1	2	3	4	5	6	7	8	9	10	11	12
δ	2	8	10	28	38	104	142	388	530	1448	1978	5404
$c_0\delta$	1	3	4	11	15	41	56	153	209	571	780	2131
$c_1\delta$		10	11	32	43	118	161	440	601	1642	2243	6128
$c_2\delta$				26	37	100	137	374	511	1396	1907	5210
$c_3\delta$						106	143	392	535	1462	1997	5456
$c_4\delta$								386	529	1444	1973	5390
$c_5\delta$										1450	1979	5408
$c_6\delta$												5402
J	$\frac{1}{120}$	$\frac{1}{160}$	$\frac{1}{120}$	$\frac{1}{105}$	$\frac{5}{456}$	$\frac{77}{6240}$	$\frac{39}{2840}$	$\frac{22}{1455}$	$\frac{7}{424}$	$\frac{311}{17376}$	$\frac{763}{39560}$	$\frac{419}{20265}$

m	13	14	15	16	17	18	19	20
δ	7382	20168	27550	75268	102818	280904	383722	1048348
$c_0\delta$	2911	7953	10864	29681	40545	110771	151316	413403
$c_1\delta$	8371	22870	31241	85352	116593	318538	435131	1188800
$c_2\delta$	7117	19444	26561	72566	99127	270820	369947	1010714
$c_3\delta$	7453	20362	27815	75992	103807	283606	387413	1058432
$c_4\delta$	7363	20116	27479	75074	102553	280180	382733	1045646
$c_5\delta$	7387	20182	27569	75320	102889	281098	383987	1049072
$c_6\delta$	7381	20164	27545	75254	102799	280852	383651	1048154
$c_7\delta$		20170	27551	75272	102823	280918	383741	1048400
$c_8\delta$				75266	102817	280900	383717	1048334
$c_9\delta$						280906	383723	1048352
$c_{10}\delta$								1048346
J	$\frac{9773}{442920}$	$\frac{28381}{1210080}$	$\frac{8213}{330600}$	$\frac{2468}{94085}$	$\frac{170393}{6169080}$	$\frac{162977}{5618080}$	$\frac{699869}{23023320}$	$\frac{8331}{262087}$

$n = 2.$

m	2	3	4	5	6	7	8	9	10	11	12
δ	3	8	60	312	155	86736	80388	522320	3457287	253598280	11937836
$c_0\delta$	1	3	21	112	55	30927	28603	186016	1230777	90294905	4250217
$c_1\delta$	4	9	76	379	192	106573	99124	643081	4259404	312347051	14705148
$c_2\delta$			46	289	132	76573	69874	457051	3016564	221544971	10423398
$c_3\delta$					172	89503	85684	549131	3656464	267523241	12607228
$c_4\delta$							76534	515161	3358804	247986521	11640078
$c_5\delta$									3528844	255093851	12084348
$c_6\delta$											11831398
J	$\frac{1}{1890}$	$\frac{11}{8960}$	$\frac{11}{12600}$	$\frac{73}{69888}$	$\frac{11}{10850}$	$\frac{134081}{124899840}$	$\frac{3961}{3617460}$	$\frac{662807}{584998400}$	$\frac{507029}{435618162}$	$\frac{3062211497}{2556270662400}$	$\frac{1028343}{835648520}$

$n = 3$.

m	2	3	4	5	6	7	8	9
δ	3	8	7248	86568	3290014	23997936	1537281648	32364150256
$c_0\delta$	1	3	2349	29392	1082811	8013897	509110987	10764281184
$c_1\delta$	4	9	9932	110209	4409946	31412443	2040010996	42647140119
$c_2\delta$			4430	76819	2225043	18665443	1105566730	24253340709
$c_3\delta$					4304484	25900993	1867200148	37040022813
$c_4\delta$							1254475462	30933891327
J	1	13	6557	61633	210047	56097271	2876254589	18892720083
	9072	17920	36529920	193912320	921203920	207342167040	11621849258880	72495696573440

3. Derivation of the formulas. Consider $\mathfrak{C}_{m,n}$. It is advantageous to translate the t-axis and to write the remainder as

$$R[x] = \int_{-\rho}^{\rho} x(t)\,dt - \sum_r e_r\,x(r) = \int_{-\rho}^{\rho} x^{(n+1)}(t)k(t)\,dt\,,$$

where $\rho = m/2$ and the index r (as also later s) ranges, throughout this paper, over the $m + 1$ values

$$-\rho, \quad -\rho + 1, \cdots, \quad \rho - 1, \quad \rho, \tag{2}$$

or indicated subsets thereof. If m is even, the range (2) consists of consecutive integers and includes zero; if m is odd, the range consists of consecutive odd halves. We treat both cases simultaneously. The coefficient e_r is the coefficient $c_{r+\rho}$ of the preceding sections.

Precisely as in [1], the relation (1) implies that

$$J = \int_{-\rho}^{\rho} k^2(t)\,dt = H + H^*\,,$$

where

$$H = \alpha - 2\sum_{r>0}\beta_r e_r + \sum_{r,s>0}\gamma_{r,s}e_r e_s\,,$$

$$H^* = \alpha - 2\sum_{r<0}\beta_{-r}e_r + \sum_{r,s<0}\gamma_{-r,-s}e_r e_s\,,$$

$$\alpha = \rho^{2n+3}/[(n+1)!^2(2n+3)]\,,$$

$$\beta_r = \int_0^r (\rho - r + t)^{n+1}t^n\,dt/[(n+1)!\,n!]\,, \qquad 0 < r,$$

$$\gamma_{r,s} = \gamma_{s,r} = \int_0^r (t + s - r)^n t^n\,dt/n!^2, \qquad 0 < r \leqq s.$$

The following lemmas provide simple expressions for the quantities β_r, $\gamma_{r,s}$.

Lemma. The quantity β_r is equal to $(-1)^{n+1}/(2n+2)!$ times the sum of the first $n+2$ terms of the binomial expansion of $(r-\rho)^{2n+2}$.

Lemma. The quantity $\gamma_{r,s}$, $r \leqq s$, is equal to $(-1)^n/(2n+1)!$ times the sum of the first $n+1$ terms of the binomial expansion of $(r-s)^{2n+1}$.

Thus, for example, if $n = 1$, $24\beta_r = r^4 - 4r^3\rho + 6r^2\rho^2$, and $6\gamma_{r,s} = -r^3 + 3r^2s$ if $r \leqq s$.

Remark. Each coefficient $\gamma_{r,s}$ is independent of ρ. The coefficients $\gamma_{r,s}$ are universal in the following sense: They occur not merely in the approximate integration here considered but also in all processes of approximation of a functional by a quantity of the form A, providing that the approximation is exact for degree n and providing that the functional is linear on the space C_n. For example, if the functional is $x(u)$, u fixed, the process is interpolation (or extrapolation) or graduation at u; if the functional is $x^{(1)}(u)$, the process is approximate differentiation; and so on.

The constraints on the e_r (because of the requirement that $A \in \mathfrak{A}_{m,n}$) are:

$$\sum_r e_r r^q = \begin{Bmatrix} 0 & \text{if } q \text{ is odd} \\ 2\rho^{q+1}/(q+1) & \text{if } q \text{ is even} \end{Bmatrix}, \qquad q = 0, \cdots, n.$$

Accordingly, as in [1], the e_r are symmetrical: $e_r = e_{-r}$; and our problem is to minimize $H = H^*$ subject to the constraints $\sum_{r>0} e_r r^q = \rho^{q+1}/(q+1)$ for all even q, if any, in $2 \leqq q \leqq n$, and

$$\sum_{r>0} e_r = \rho \qquad \text{if } m \text{ is odd},$$

$$e_0 + 2\sum_{r>0} e_r = 2\rho \qquad \text{if } m \text{ is even}.$$

For example, in $\mathfrak{A}_{3,1}$,

$$H = 243/640 - 86e_{1/2}/384 - 162e_{3/2}/128 + e_{1/2}^2/24 + e_{1/2}e_{3/2}/3 + 9e_{3/2}^2/8,$$

and there is one constraint:

$$e_{1/2} + e_{3/2} = 3/2.$$

The minimization may be carried out directly or by the method of Lagrange's multipliers. The results are

$$e_{1/2} = 1.1, \qquad e_{3/2} = 0.4, \qquad H = 1/240 = J/2.$$

4. The case $n = 1$. There are striking recursions in the entries of the table of §2, $n = 1$. For example, minus any entry plus four times the entry two places on its right equals the entry four places on its right. This is a consequence of (8) below.

In this section we consider $n = 1$ only. The index ρ ranges over the values $1/2, 1, 3/2, 2, \cdots$. Denote the coefficients e_r of the best formula in $\mathfrak{A}_{2\rho,1}$ by $e_{r,\rho}$. The index r has the range (2). Put

$$\nu_{-3/2,-3/2} = -1, \qquad \nu_{-1,-1} = 1, \qquad \nu_{-1/2,-1/2} = 0, \qquad \nu_{0,0} = 1; \tag{3}$$

$$\nu_{\rho,\rho} = 4\nu_{\rho-1,\rho-1} - \nu_{\rho-2,\rho-2} = \nu_{-\rho,\rho},$$

$$\delta_\rho = 2(\nu_{\rho,\rho} + \nu_{\rho-1,\rho-1}), \qquad \delta_0 = 2(\nu_{0,0} + \nu_{-1,-1}) = 4, \tag{4}$$

$$\nu_{r,\rho} = \delta_\rho + (-1)^{\rho-r+1}\delta_r/2 = \nu_{-r,\rho} \qquad \text{if } 0 \leqq r < \rho.$$

Then

$$e_{r,\rho} = \nu_{r,\rho}/\delta_\rho \; ; \tag{5}$$

so that the entire table for $n = 1$ is determined by (3), (4).

The relations (3), (4) permit us to deduce explicit forms for $\nu_{\rho,\rho}$ and δ_ρ (and hence for $\nu_{r,\rho}$):

$$\left.\begin{aligned} \nu_{\rho,\rho} &= [(3+\sqrt{3})(2+\sqrt{3})^\rho + (3-\sqrt{3})(2-\sqrt{3})^\rho]/6 \\ \delta_\rho &= 2[(2+\sqrt{3})^\rho + (2-\sqrt{3})^\rho] \end{aligned}\right\} \quad \rho = 1, 2, \cdots,$$

$$\left.\begin{aligned} \nu_{\rho,\rho} &= [\sqrt{3}(2+\sqrt{3})^{\rho+1/2} - \sqrt{3}(2-\sqrt{3})^{\rho+1/2}]/6 \\ \delta_\rho &= (-1+\sqrt{3})(2+\sqrt{3})^{\rho+1/2} - (1+\sqrt{3})(2-\sqrt{3})^{\rho+1/2} \end{aligned}\right\} \rho = 1/2, 3/2, \cdots. \tag{6}$$

One may establish the validity of these formulas by showing directly that $e_{r,\rho}$ defined by (5), (6) satisfy the appropriate constraints and minimizing equations. Such a demonstration, though long, is straightforward; and we have carried it through. One may alternatively work with (5), (3), and (4) and proceed inductively.

Let J_ρ be the value of J for the best formula in $\mathfrak{C}_{2\rho,1}$. The following recursion is satisfied for $3/2 \leqq \rho \leqq 10$:

$$J_\rho - J_{\rho-1} = \nu_{\rho,\rho}\nu_{\rho-2,\rho-2}/(15\delta_\rho\delta_{\rho-1}). \tag{7}$$

It would be a straightforward but elaborate calculation to decide whether this relation holds for all $\rho \geqq 3/2$.

The relations (4) imply others, among which we note the following:

$$\delta_\rho = 4\delta_{\rho-1} - \delta_{\rho-2}, \qquad \delta_{-1/2} = -2, \qquad \delta_{-3/2} = -10, \qquad \delta_{-1} = 8,$$
$$\nu_{r,\rho} = 4\nu_{r-1,\rho-1} - \nu_{r-2,\rho-2} \qquad \text{if } r \geqq -\rho + 5, \tag{8}$$

$$\delta_\rho = [3 + (-1)^{2\rho}]\delta_{\rho-1/2}/2 + \delta_{\rho-1},$$
$$\nu_{r,\rho} = [3 + (-1)^{2\rho}]\nu_{r-1/2,\rho-1/2}/2 + \nu_{r-1,\rho-1} \quad \text{if } r \geqq -\rho + 3. \tag{9}$$

The cases m even and m odd can be combined in the following way. Put

$$\delta^*_\rho = \begin{cases} \delta_\rho \\ \sqrt{2}\delta_\rho \end{cases}, \qquad \nu^*_{r,\rho} = \begin{cases} \nu_{r,\rho} & \text{if } \rho \text{ is integral,} \\ \sqrt{2}\nu_{r,\rho} & \text{if not.} \end{cases}$$

Then $e_{r,\rho} = \nu^*_{r,\rho}/\delta^*_\rho$; and relations (4) and (7), with ν and δ starred, remain true. Relations (9), (4), and (3) imply that

$$\delta^*_\rho = \sqrt{2}\delta^*_{\rho-1/2} + \delta^*_{\rho-1},$$
$$\nu^*_{\rho,\rho} = \sqrt{2}\nu^*_{\rho-1/2,\rho-1/2} + \nu^*_{\rho-1,\rho-1},$$
$$\delta^*_\rho = 4\nu^*_{\rho,\rho} - 2\sqrt{2}\nu^*_{\rho-1/2,\rho-1/2}.$$

Relations (6) simplify:

$$\nu^*_{\rho,\rho} = [(1+\sqrt{3})^{2\rho+1} - (1-\sqrt{3})^{2\rho+1}]/(\sqrt{3}\,2^{\rho+1}),$$
$$\delta^*_\rho = [(1+\sqrt{3})^{2\rho} + (1-\sqrt{3})^{2\rho}]/2^{\rho-1}. \tag{10}$$

5. Convergence of coefficients; unsolved problems. Consider a fixed n. Let the coefficients e_r and the J of the best formula in $\mathfrak{C}_{2\rho,n}$ be denoted by $e_{r,\rho}$ and J_ρ, respectively. The best formulas in $\mathfrak{C}_{2\rho,n}$ seem to have the following properties:

(i) For fixed positive integer or half-integer r, the sequence $e_{r,\rho}$, $\rho = r, r+1, r+2, \cdots$, converges to 1.

(ii) For fixed non-negative integer i, the sequence $e_{\rho-i,\rho}$, $\rho = i/2, (i+1)/2, (i+2)/2, \cdots$, converges.

(iii) The sequences $J_\rho - J_{\rho-1/2}$ and (hence) $J_\rho - J_{\rho-1}$, $\rho = 3/2, 2, 5/2, 3, \cdots$, converge.

For $n = 0$, all three properties are trivial, since $e_{\rho,\rho} = e_{-\rho,\rho} = 1/2$, all other $e_{r,\rho} = 1$, and $J_\rho = \rho/6$. (See [1].)

For $n = 1$, properties (i) and (ii) are immediate consequences of (10) and (4). Indeed, $\lim e_{r,\rho} = 1$ for fixed r; $\lim e_{\rho,\rho} = (3+\sqrt{3})/12$; and $\lim e_{\rho-i,\rho} = 1 + (-1)^{i+1}(2-\sqrt{3})^i/2$ for fixed integral $i > 0$. If (7) holds for all $\rho \geqq 3/2$, then it will follow that $\lim (J_\rho - J_{\rho-1}) = 1/360$. From the table of §2 it appears also that $\lim (J_\rho - J_{\rho-1/2}) = 1/720$.

For $n \geqq 2$, these questions are still open. The table of §2, expressed in decimal form, constitutes heuristic evidence that (i), (ii), and (iii) are true.

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PRANDTL-MEYER FLOW BEHIND A CURVED SHOCK WAVE

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1. Introduction. It has been shown[1] that if a plane uniform supersonic stream encounters a curved compression shock, the flow downstream of the shock cannot be irrotational. It follows that the two-dimensional supersonic flow behind a shock wave attached to a profile cannot be of the Prandtl-Meyer type, although the assumption of the Prandtl-Meyer regime is often made for lack of more accurate methods. This paper presents an indirect method of measuring the error due to such an assumption; namely, to construct a free stream which is compatible with Prandtl-Meyer flow and a curved shock wave, then to compare this free stream with a uniform stream.

In Section 2 a Prandtl-Meyer flow about a profile is assumed to be the deflected flow behind a curved shock wave attached to the profile. In Section 3 a free stream compatible with this regime is constructed, and in Section 4 the curved shock separating the two flows is determined, its equation found, and some of its properties examined. The final section is devoted to a quantitative analysis of the velocity and stagnation pressure of the constructed stream. It is shown that for sufficiently sharp profiles the constructed stream differs only slightly from a uniform stream, but for blunt profiles and high Mach numbers the non-isentropic character of the constructed free stream is very marked.

We assume initially a profile introduced into a supersonic stream which has constant velocity and entropy in the neighborhood of the profile. For the profile itself we choose the simplest case of a semi-wedge with a face of finite length terminated at a sharp corner by a plane parallel to the original flow. We further assume that the semi-wedge angle and the free stream velocity are so chosen that an attached shock wave results, with supersonic flow along the wedge face. It is well known that in this case the shock wave is curved, and we assume the Prandtl-Meyer regime to exist everywhere behind the curved shock, the gas possessing those properties necessary for this regime. We restrict ourselves to the case of $\gamma = 7/5$.

The shock wave profile for this flow will be linear near the wedge, becoming curved beyond the point where it is influenced by the expansion. We therefore divide the free stream into two regions (see Fig. 1), region C upstream of the linear portion of the shock, and region V upstream of the curved portion. The velocity and entropy are constant in region C and vary, as will be shown, in region V. The flow behind the shock wave we divide into three regions, region 1 being the region of constant flow parallel to the wedge face, region 2 being that in which the expansion about the corner takes place, and region 3 being that in which the flow is again constant and parallel to the original flow. The exact boundaries of these regions will be defined later.

[1] Sauer, R., Theoretische Einführung in die Gasdynamik, Springer-Verlag, Berlin 1943, Section 20, Par. 1.

The variables used to describe the flow are: stagnation pressure, p_0, and "velocity number" M^* defined by the relations

$$M^{*2} = \frac{6M^2}{5 + M^2}; \qquad \sin^2 \mu = \frac{1}{M^2} = \frac{6 - M^{*2}}{5M^{*2}}, \tag{1}$$

where μ is Mach angle, M is Mach number, and M^* may be recognized as the ratio of gas velocity to the critical velocity of sound, thus being dimensionless and proportional to velocity. It can be seen from (1) that M^* varies monotonically from 1 to $\sqrt{6}$ as M varies from 1 to ∞. We shall denote the values of these variables in different regions by the subscripts C, V, 1, and 3; omitting subscripts in region 2. Thus the quantities M_C^*, M_1^*, and M_3^* are constant, but

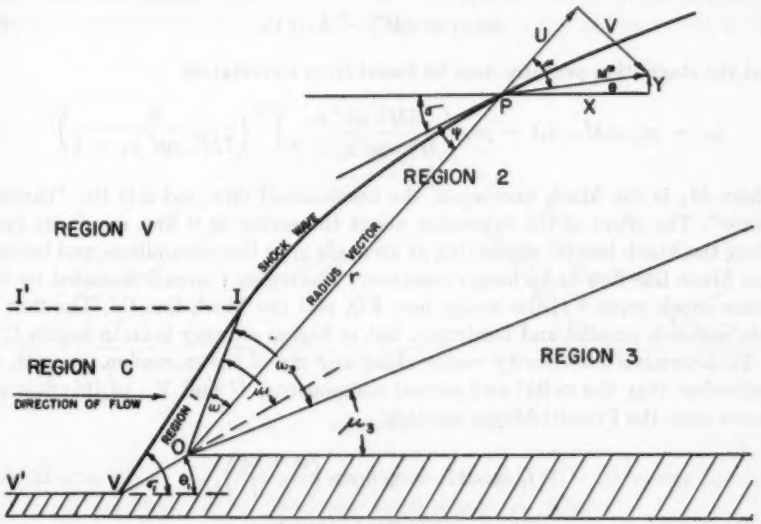


FIGURE 1 PHYSICAL PLANE

M_V^* and M^* are variable. Mach angles will be denoted by μ and the shock angle by σ. By the term "velocity vector" we shall mean the vector with polar coordinates (M^*, θ), where M^* is given by (1) and θ is the inclination of the flow to the direction of the original free stream.

2. The flow behind the shock. Referring to Fig. 1, we have a uniform flow along $V'V$ encountering a semi-wedge of angle θ_1 at V, which causes an attached shock wave of angle σ_1. The velocity number of the gas in region C is M_C^*, and the stagnation pressure p_{0C}, both quantities being given in advance. We proceed now to the determination of the flow behind the shock.

To determine the velocity vector of the flow parallel to VO, we refer to the "shock polar" equation which the rectangular components, X and Y, of the de-

flected flow must satisfy; namely,[2]

$$Y^2 = \frac{(M_c^* - X)^2(M_c^* X - 1)}{1 - M_c^* X + (5/6)M_c^{*2}}. \tag{2}$$

The flow parallel to VO is then determined when we add the condition

$$Y = X \tan \theta_1, \tag{3}$$

which requires that the deflected flow be parallel to the wedge face. The common solution (X_1, Y_1) of (2) and (3) then determines the rectangular components of the velocity vector (M_1^*, θ_1) which describes the flow in region 1. The shock angle σ_1 is given by

$$\tan \sigma_1 = (M_c^* - X_1)/Y_1, \tag{4}$$

and the stagnation pressure may be found from the relation

$$p_{0_1} = p_{0_C} z(M_C, \sigma_1) = p_{0_C} \left(\frac{6M_C^2 \sin^2 \sigma_1}{M_C^2 \sin^2 \sigma_1 + 5}\right)^{7/2} \left(\frac{6}{7M_C^2 \sin^2 \sigma_1 - 1}\right)^{5/2} \tag{5}$$

where M_C is the Mach number of the undisturbed flow and z is the "throttle factor". The effect of the expansion about the corner at 0 first manifests itself along the Mach line OI emanating at an angle μ_1 to the streamlines, and beyond this Mach line flow is no longer constant. The region 1 then is bounded by the linear shock wave VI, the wedge face VO, and the Mach line OI. The flow in this region is parallel and isentropic, but of higher entropy than in region C.

To determine the velocity vector along any ray of the expansion fan at 0, we remember that the radial and normal components, U and V, of this flow are found from the Prandtl-Meyer solution

$$\begin{aligned} U &= M_r^* = \sqrt{6} \sin\left(\frac{\omega + \omega_1}{\sqrt{6}}\right), && \omega > 0, \\ V &= M_\theta^* = \cos\left(\frac{\omega + \omega_1}{\sqrt{6}}\right), && \omega > 0, \end{aligned} \tag{6}$$

where ω is measured clockwise from OI and ω_1 is a constant satisfying the relations

$$\sin \frac{\omega_1}{\sqrt{6}} = \frac{M_1^*}{\sqrt{6}} \cos \mu_1 = \frac{\sqrt{M_1^{*2} - 1}}{\sqrt{5}},$$

$$\cos \frac{\omega_1}{\sqrt{6}} = M_1^* \sin \mu_1 = \frac{\sqrt{6 - M_1^{*2}}}{\sqrt{5}}.$$

In the hodograph plane, the magnitude M^* and the direction θ of the velocity vector for clockwise expansion are connected by the equation of the clockwise

[2] For a discussion of equations (2) through (7) see Miles, E. R. C., Supersonic Aerodynamics, McGraw-Hill, 1950, Chapters 7 and 4.

epicycloid

$$\theta = \theta_1 - \sqrt{6}\arctan\frac{(\beta-\beta_1)}{\sqrt{6}} + \arctan(\beta-\beta_1)$$
$$= -\sqrt{6}\arctan\left(\frac{\beta-\beta_3}{\sqrt{6}}\right) + \arctan(\beta-\beta_3), \tag{7}$$

where

$$\beta = \cot\mu = \frac{\sqrt{6}\sqrt{M^{*2}-1}}{\sqrt{6}-M^{*2}}, \qquad \beta_1 = \beta(M_1^*), \quad \beta_3 = \beta(M_3^*),$$

and the second form of (7) determines β_3 and hence M_3^*, which is the velocity number of the constant, parallel flow after the expansion about 0 has been completed. Geometrically, in Fig. 2, M_2^* is represented by the radius vector to the

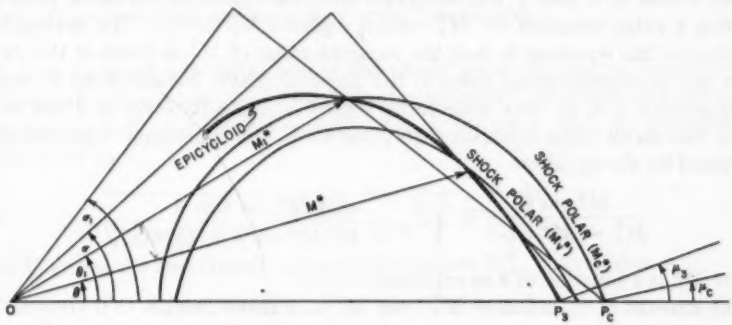


Figure 2 HODOGRAPH PLANE

point P_3 at which the clockwise epicycloid through (M_1^*, θ_1) meets the line $\theta = 0$, and formulae (6) are then valid for $0 < \omega < \omega_3$, where

$$\omega_3 = \theta_1 + \mu_1 - \mu_3. \tag{8}$$

The inclination of the streamlines along any ray from 0 in Fig. 1 may be found from the relation

$$\theta = \theta_1 + \mu_1 - \omega - \mu \tag{9}$$

where

$$\mu = \arctan\frac{V}{U} = \arctan\frac{1}{\sqrt{6}}\cot\left(\frac{\omega+\omega_1}{\sqrt{6}}\right). \tag{10}$$

If we wish to compute the pressure p in region 2 we make use of Bernoulli's equation since the expansion is adiabatic and have

$$\frac{p}{p_0} = \left(\frac{6-M^{*2}}{6}\right)^{7/2}, \tag{11}$$

where $p_0 = p_{0_1}$, as given by (5). The final pressure in region 3 is constant and may be found by putting $M^* = M_3^*$ in (11).

3. Construction of a compatible free stream. Having determined the flow downstream of the shock wave, we are faced by the problem of determining a shock curve and free stream consistent with this flow. In the hodograph plane the velocity vectors of the flow behind the shock are represented by the radius vectors to the epicycloid for $0 < \theta < \theta_1$. But if a uniform stream encounters a shock wave, the possible velocity vectors of the flow immediately behind the shock are represented by the radius vectors to a single member of the shock polar family (2). Hence it is evident that the free stream necessary here is not uniform, but must be determined as follows: For each point of the epicycloid (corresponding to some point immediately behind the shock wave in the physical plane) values of X and Y are computed and substituted in the shock polar (2) forming a cubic equation for M_V^* (which replaces M_C^* in (2)). The appropriate solution to this equation is then the required value of M_V^* in front of the shock wave at the corresponding point in the physical plane. Inasmuch as X and Y are functions of θ, we may henceforth regard M_V^* as a function of θ and if we throw the shock polar equation into polar form this functional relationship is expressed by the equation

$$\frac{M^* \sin\theta}{M_V^* - M^* \cos\theta} = \sqrt{\frac{M_V^* M^* \cos\theta - 1}{1 - M_V^* M^* \cos\theta + (5/6) M_V^{*2}}}, \tag{12}$$

where M^* is a function of θ as expressed in (7).

The amount of variation of M_V^* from M_C^* as θ varies from θ_1 to 0 gives an indirect measure of the amount of error introduced by the assumption of Prandtl-Meyer flow if the free stream is uniform. It can be seen that the study of the function M_V^* is a difficult one mathematically, but we can obtain certain pertinent results. First we know the velocity number M_C^* is consistent with the Prandtl-Meyer flow at I (Fig. 1), hence we have $M_V^* = M_C^*$ at $\theta = \theta_1$. It is clear also that for $\theta = 0$, $M_V^* = M_3^*$, and the geometrical representation of $M_C^* - M_3^*$ is given in Fig. 2 by the segment P_3P_C. Near $\theta = 0$ we can obtain some additional information, as given in

THEOREM I: *The first and second derivatives of the function* $M_V^*(\theta)$ *defined by* (12) *both vanish at the point* $(M_3^*, 0)$.

To prove this theorem we recall[3] that if in the hodograph plane we have an epicycloid of the family (7) and a shock polar of the family (2) coinciding at the double point of the shock polar, then the slopes and curvatures of the two curves also coincide. To interpret this for our purposes, let us suppose (12) and (7) were written in the forms:

$$S(M_V^*, M^*, \theta) = 0, \tag{12'}$$

$$M^* = E(\theta). \tag{7'}$$

[3] Sauer, Section 18, Par. 2.

If we substitute from (7′) into (12′) to obtain $M_\nu^*(\theta)$ and then differentiate, we find

$$\frac{dM_\nu^*}{d\theta} = -\frac{1}{\partial S/\partial M_\nu^*}\left(\frac{\partial S}{\partial M^*}\frac{dE}{d\theta} + \frac{\partial S}{\partial \theta}\right).$$

Now at $\theta = 0$ we know $dE/d\theta = dM^*/d\theta$ if M_ν^* is fixed, and if we differentiate (12′) holding M_ν^* fixed we see that the right member of the last equation vanishes. A similar argument for the second derivative will show that if $d^2E/d\theta^2 = d^2M^*/d\theta^2$, then $d^2M_\nu^*/d\theta^2 = 0$, and the theorem is proved.

If we wish to represent M_ν^* as a function of θ in the hodograph plane, this theorem tells us that for small θ the curves are closely approximated by circles of radii M_3^* with center at the origin. The curves $M_\nu^* = M_\nu^*(\theta)$ will terminate in the hodograph plane when the value of θ becomes too large to support an attached shock for the corresponding value of M_ν^*. Curves of M_ν^* vs. θ with M_3^* as parameter are given in Fig. 3, using rectangular coordinates.

4. The equation and properties of the shock wave. To obtain the equation of the shock wave we use a polar coordinate system with 0 as pole and the angle ω as independent variable. Referring to Fig. 1, we have, with the distance OP represented by r,

$$\frac{1}{r}\frac{dr}{d\omega} = \cot \Psi = \frac{\cos(\theta + \mu - \sigma)}{\sin(\theta + \mu - \sigma)}. \tag{13}$$

Recalling that the free stream has velocity number M_ν^*, the relation

$$\tan \sigma = (M_\nu^* - M^* \cos \theta)/M^* \sin \theta$$

when used in (13) yields for the equation of the shock wave

$$\log \frac{r}{r_1} = \int_0^\omega \frac{M_\nu^* \sin(\theta + \mu) - V}{U - M_\nu^* \cos(\theta + \mu)}\, d\omega, \tag{14}$$

where r_1 is the length of the segment OI and is finite, as will be shown below. In the integrand, U and V are known functions of ω from (6), θ and μ are functions of ω as given by (9) and (10), and M_ν^* has been shown to be a function of θ, hence also of ω. We can prove the following properties of the shock wave:

a. *The point I at which the shock wave begins to curve is in the finite plane for all $\theta_1 > 0$.* To prove this we need show only that the Mach line OI of the deflected flow is not parallel to the shock wave VI. But if these two lines were parallel, it is easily shown that the Mach lines of the free stream would also be parallel to the shock wave, which is impossible for non-degenerate shocks. Hence an intersection is assured, and if the deflected stream is supersonic, it must be of the type shown in Fig. 1.

b. *The shock wave degenerates into a Mach line for $\omega = \omega_3$.* For $\omega = \omega_3$, $\mu = \mu_3$ and we see from (9) that $\theta = 0$ and hence that $M_\nu^* = M_3^*$. Since this is the velocity number of the stream in region 3, the stream is unchanged by the shock and the shock is a Mach line.

The question remains as to whether $r(\omega_3)$ is finite; i.e., whether the shock degenerates in the finite plane. This question we answer with

THEOREM II: *The distance* $r(\omega)$ *as given by* (14) *becomes infinite as* $\omega \to \omega_3$.

The proof requires first the

Lemma: *The functions*

$$N(\omega) = \frac{M_V^* \sin(\theta + \mu) - V}{M_3^* \sin(\theta + \mu) - V} \text{ and } D(\omega) = \frac{U - M_V^* \cos(\theta + \mu)}{U - M_3^* \cos(\theta + \mu)}$$

are continuous at $\omega = \omega_3$ *and their limit is unity.*

Since M_V^*, $\theta + \mu$, U, and V are all analytic functions of ω at $\omega = \omega_3$ (where $\theta = 0$), and since reference to Fig. 1 shows N and D to be indeterminate at $\omega = \omega_3$, we differentiate numerator and denominator of N to obtain, using (9),

$$\lim_{\omega \to \omega_3} N = 1 + \lim_{\omega \to \omega_3} \frac{(dM_V^*/d\omega) \sin(\theta + \mu)}{-M_3^* \cos(\theta + \mu) - (dV/d\omega)}.$$

But from (6) we see that $dV/d\omega = -U/6$, and at $\theta = 0$ the radial component U has the value of $M_3^* \cos \mu_3$. Hence the denominator does not vanish, and from Theorem I we conclude that

$$\lim_{\omega \to \omega_3} N = 1.$$

In evaluating D we have

$$\lim_{\omega \to \omega_3} D = \lim_{\omega \to \omega_3} \left(N + \frac{(dM_V^*/d\theta) \cos(\theta + \mu)}{V - M_3^* \sin(\theta + \mu)} \right)$$

and the quotient is still indeterminate. A second application of L'Hopital's rule produces the desired result.

To prove the theorem we remember that for non-degenerate shocks $\theta + \mu > \sigma$, so that the quotient in (13), and hence the integrand in (14), is positive for $\omega < \omega_3$. Using the Lemma, we can choose some value of $\omega < \omega_3$, say $\hat{\omega}$, such that $N/D > 1/2$ for $\hat{\omega} < \omega < \omega_3$. Then from (14), we can write for $\omega > \hat{\omega}$

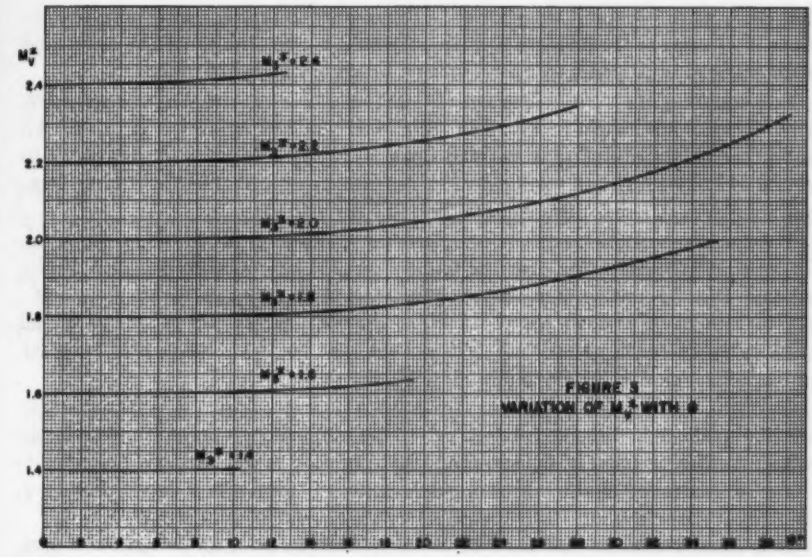
$$\log \frac{r}{r_1} > \int_0^{\hat{\omega}} \frac{M_V^* \sin(\theta + \mu) - V}{U - M_V^* \cos(\theta + \mu)} d\omega + \frac{1}{2} \int_{\hat{\omega}}^{\omega} \frac{M_3^* \sin(\theta + \mu) - V}{U - M_3^* \cos(\theta + \mu)} d\omega.$$

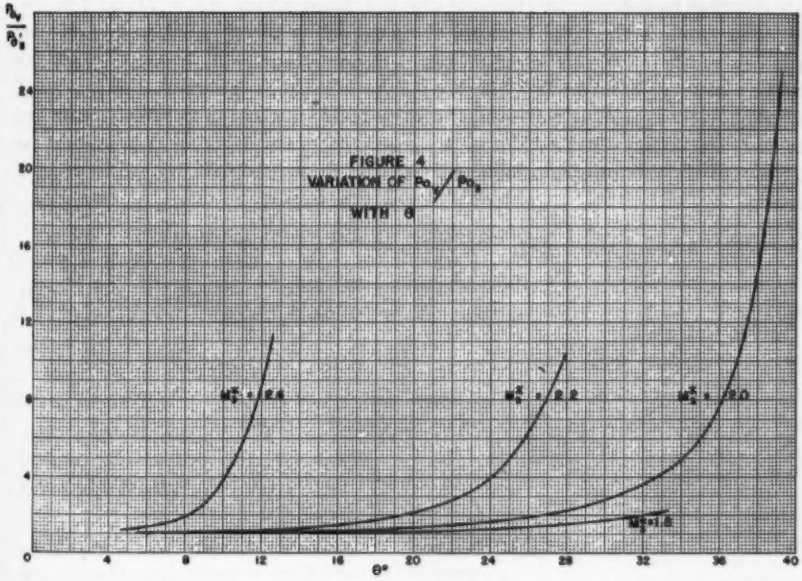
The second integral on the right may be evaluated with the aid of (6) and we obtain

$$\log \frac{r}{r_1} > \log \left(\frac{\hat{r}}{r_1} \right) + \tfrac{1}{2} \log \left(\frac{\hat{U} - M_3^* \cos(\hat{\mu} + \hat{\theta})}{U - M_3^* \cos(\mu + \theta)} \right),$$

where $\hat{r}$, $\hat{\theta}$, $\hat{\mu}$, and $\hat{U}$ are the values of these variables at $\omega = \hat{\omega}$. The second term on the right is clearly unbounded in the vicinity of ω_3, and the theorem is proved.

5. Comparison of the constructed free stream with an isentropic free stream. In an analysis of constructed free streams, we are apparently faced with two independent parameters, θ_1 and M_C^*. We can reduce these two parameters to a





single one by observing, in Fig. 1, that if the flow is determined for fixed M_C^* and maximum θ_1, then all flows having the same value of M_3^* can be obtained by moving the point I along the shock wave. At any point, M_V^* can be taken as a

new M_C^*, with θ (behind the shock) as the corresponding θ_1. This becomes obvious if we refer to the hodograph plane, where it is clear that once the epicycloid is determined, any point (M_1^*, θ_1) can be taken as the initial point.

In comparing M_V^* with M_C^* we use (12) to plot M_V^* vs. θ with M_3^* as parameter. These graphs are shown in Fig. 3 in which the value of M_C^* is given as $M_V^*(\theta_1)$, and other values of M_V^* may be read for $\theta < \theta_1$. In fact, we can use Fig. 3 to obtain M_V^* along any line parallel to $V'V$ by measuring ω at the shock wave, computing θ from (9) and (10), then reading M_V^* from the graph.

We can use a similar parameterization for stagnation pressures in region V, for once M_V^* is known, p_{0_V} may be found from (5) by replacing p_{0_C} and M_C by p_{0_V} and M_V. The value of p_{0_1} is, of course, the same as p_{0_3}, and for given M_3^*, p_{0_V}/p_{0_3} can be plotted as shown in Fig. 4. For $M_3^* \leqq 1.6$, $p_{0_V}/p_{0_3} \leqq 1.1$ for all possible θ, but for large M_3^* and θ it is seen that total variation of stagnation pressure in region V is very large.

In Fig. 3 and Fig. 4 the slopes of the curves are so small near $\theta = 0$ that quantitative analysis is difficult for $\theta < 10°$. This phenomenon is, of course, closely related to the problem of determining the "pressure series" for $\Delta p/q$ as a function of θ. Inasmuch as the radius of convergence of this series is as yet unknown, we cannot make a quantitative analysis even for "sufficiently small" θ.[4]

It is clear, however, that for $\theta > 10°$ the Mach number and stagnation pressure decrease monotonically from the line $V'V$ along which $M_V^* = M_C^*$ and $p_{0_V} = p_{0_C}$. For small θ, the total decrease is relatively small and the Prandtl-Meyer flow behind the shock may be considered in good agreement (except for effects due to viscosity) with the flow produced by a uniform free stream. For values of θ near shock detachment, however, the value of M_V^* and p_{0_V} in the free stream are seriously at variance with M_C^* and p_{0_C}, and the Prandtl-Meyer flow cannot be considered an accurate approximation to the flow produced by an isentropic stream.

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[4] Miles, Chapter 11.

DETERMINATION OF AXIALLY SYMMETRIC FLOW PATTERNS OF A COMPRESSIBLE FLUID[1,2]

By S. Bergman

Introduction. The potential function of an axially symmetric flow of a compressible fluid satisfies a non-linear partial differential equation, whose mathematical theory is comparatively unknown. On the other hand, the corresponding functions in the two-dimensional case, considered in the hodograph plane, satisfy a linear equation, for which a number of flow patterns have been obtained.

The analogy to the plane case suggests that we consider the equation for the potentials of axially symmetric flows. Then one can use the solution $\Phi^{(0)}$ of the corresponding equation for a plane flow as a first approximation for the function Φ connected in a simple manner with the above-mentioned potential function. One then obtains for the function $\Phi - \Phi^{(0)}$ a non-linear equation. Applying the method of successive approximations, we replace the equation for $\Phi - \Phi^{(0)}$, see (1.2), by a sequence of linear non-homogeneous equations $L_n(\Phi^{(n)}) = 0$, see (1.5).

An important task which arises is to develop procedures suitable for practical purposes for solving these equations, at least for small values of n, say $n = 1$ and 2. In the study of a linear equation, say L, the fact that if ψ_ν, $\nu = 1, 2, \cdots$ are solutions of L, the linear combination $\Sigma_\nu c_\nu \psi_\nu$, where c_ν are arbitrary constants, is also a solution of L, is of great advantage for numerical applications. This permits us, for instance, to prepare, once and for all, tables of a set of particular solutions ψ_ν which can then be used conveniently in determining a solution satisfying prescribed initial or boundary conditions. As we show in the present paper, certain tables can be used for the computation of at least the first and second approximations $\Phi^{(1)}$ and $\Phi^{(2)}$.

In the present paper, we limit ourselves to the study of analytic solutions of the equation for $\Phi - \Phi^{(0)}$. The advantage in making this restriction and in considering the equation in the complex domain is that we can simultaneously consider the sub- and supersonic regions and obtain solutions for the mixed case.

In §1 we derive the equation for the set of solutions of equations (1.5). We show that in a sufficiently small neighborhood N_ρ the equations (1.5) can be approximated by another set of equations, [see (1.7)], for which it is possible to prepare auxiliary tables, as mentioned above. The procedure suggested requires dividing the domain, say N, in which the solution $\Phi^{(0)}$ is considered, into a

[1] Research paper done under Navy Contract NOrd 10-449, Task 3, at Harvard University. The ideas expressed in this paper represent the personal view of the author and are not necessarily those of the Bureau of Ordnance.

[2] The present paper will also appear as Report No. 16 of the series "Operator Methods in the Theory of Compressible Fluids," Harvard University, Division of Engineering Sciences, where an appendix with auxiliary formulas will be incorporated. We shall refer to it as "Appendix."

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number of neighborhoods N_μ. In §2 we discuss the case where a neighborhood N_μ includes the sonic line, while §3 and §4 are devoted to the cases where N_μ lies entirely in the sub- or supersonic region. In §5, we discuss how to combine solutions Φ_μ, each of which is defined in the neighborhood N_μ, into one solution defined in the domain $N = \Sigma N_\mu$. §6 is devoted to considerations of more theoretical character pertaining to the question of the convergence of the approximation functions when $n \to \infty$.

I wish to express my thanks to Mr. Henry Pollak for his assistance in the preparation of the present paper, in particular for many helpful suggestions.

1. The equation of axial symmetric flow. The method of infinite approximation by a sequence of linear equations. Let x, r and χ denote the cylindrical coordinates in the physical plane (i.e., in the plane in which the flow actually takes place) and let w, θ, and χ' denote the spherical coordinates in the hodograph plane. Then *the quantity*

$$\Phi = x\varphi_x + r\varphi_r - \varphi \tag{1.1}$$

where φ is the potential function, satisfies the equation

$$L(\Phi) = N(\Phi), \tag{1.2}$$

where

$$L(\Phi) = w^{-2}(w^2\Phi_{ww} + w(1 - w^2c^{-2})\Phi_w + (1 - w^2c^{-2})\Phi_{\theta\theta}), \tag{1.2a}$$

$$N(\Phi) = \Phi_v^{-1}(w^{-1}\sin\theta\,[\Phi_{ww}(w\Phi_w + \Phi_{\theta\theta}) - (\Phi_{w\theta} - w^{-1}\Phi_\theta)^2]), \tag{1.2b}$$

Here $c^2 = k(\hat{w}^2 - w^2)$ and $\Phi_v = \Phi_w \sin\theta + w^{-1}\Phi_\theta \cos\theta$, $k = \frac{1}{2}(\gamma - 1)$, $\hat{w}$(= maximum speed) being constants. γ is the gas constant. (See e.g. [12][3] p. 138.)

In that part of the plane where r (the distance of the flow under consideration from the rotation axis) is substantially larger than zero, one can consider a solution for the corresponding equation

$$L(\Phi^{(0)}) = 0 \tag{1.3}$$

for a plane flow as a first approximation to a solution of the equation (1.2). Here

$$\Phi^{(0)} = x\varphi_x^{(0)} + r\varphi_r^{(0)} - \varphi^{(0)} \tag{1.4}$$

where x, r now denote the Cartesian coordinates in the plane, and $\varphi^{(0)}$ is the complex potential. (See [12] p. 95.)

The method to be discussed in the first part of the present paper is based on the fact that for those $\varphi^{(0)}(x, r)$ which yield flow patterns around profiles of interest in applications, comparatively simple formulas can be obtained, using, for instance, the operator method [5, 6, 7, 8, 9], or developments in series of hypergeometric functions. If then the functions $\Phi^{(n)}$ obtained by solving the set of equations

$$L(\Phi^{(n)}) = N(\Phi^{(n-1)}), \qquad n = 1, 2, \cdots \tag{1.5}$$

[3] Numbers in brackets refer to the bibliography (p. 145).

converge to a limit Φ, then in many instances Φ will represent a solution of equation (1.1) with slightly distorted boundary conditions. The corresponding

$$\varphi = w\Phi_w - \Phi \tag{1.6}$$

will represent a potential (in the hodograph plane) of an axial symmetric flow around a profile which is obtained by a distortion from the profile corresponding to $\varphi^{(0)}$. By some additional considerations these distortions can be diminished considerably. If $\Phi^{(0)}$, the determination of which is discussed below, is known, then the first question that arises from a practical point of view is in what manner the equations (1.5) can be solved at least for small values of n.

If we are dealing with a linear homogeneous equation, the possibility of determining a complete set of particular solutions is a great advantage. This permits us to prepare once and for all tables of particular solutions which can then be used conveniently in computing the desired solution, for those values of the arguments in which we are interested; in addition, it gives us a certain insight into the variation of the solution if some parameters, for instance the boundary data, vary.

It is desirable in dealing with non-linear equations to work out procedures as far as possible where tables can be determined for the computation of successive functions $\Phi^{(n)}$. In our special case this can be done if the expression $N(\Phi^{(n-1)})$ is replaced by

$$N_0(\Phi^{(n-1)}) = -\frac{2v}{\Phi_{v,\max}^{(n-1)} + \Phi_{v,\min}^{(n-1)}} \frac{\partial(\Phi_w^{(n-1)}, \Phi_v^{(n-1)})}{\partial(u, v)} \tag{1.7}$$

where $\Phi_{v,\max}^{(n-1)}$ and $\Phi_{v,\min}^{(n-1)}$ are the maximum and minimum values of $\Phi_v^{(n-1)}$ in the domain in which the function is considered. In §3, we shall discuss the application of such a procedure in our case.

A method for determining $\Phi^{(0)}$ may be briefly described as follows: By separation of variables, we obtain solutions of (1.3) of the form

$$\Phi^{(0)} = \sum_{\nu=-N}^{N} w^{|\nu|} B_\nu e^{i\nu\theta} F_{|\nu|}, \tag{1.8}$$

where

$$\begin{aligned} F_{|\nu|} &= \mathrm{F}((1/2)\nu\hat{w}^2 + (1/4k) + A, (1/2\nu)\hat{w}^2 + 1/(4k) - A, \nu + 1, w^2\hat{w}^{-2}) \\ 4A^2 &= \nu^2\hat{w}^4 + k^{-1}(\nu\hat{w}^2 + \nu^2 - \nu) + (4k^2)^{-1}. \end{aligned} \tag{1.9}$$

In order to determine the B_ν corresponding to the two-dimensional flow around a given profile, we can first find the stream function $\psi^{(0)}$ of such a flow by means of the method of integral operators (see §4 of [6] and §2 of [8]). Then using (1.6), and (173) of [12], we obtain

$$\begin{aligned} w\Phi_{ww}^{(0)} &= -\rho_0(\rho w)^{-1}(1 - w^2c^{-2})\psi_\theta^{(0)} \\ w\Phi_{w\theta}^{(0)} &= \Phi_\theta^{(0)} = \rho_0\rho^{-1}w\psi_w^{(0)} \end{aligned} \tag{1.10}$$

where

$$\rho = \rho(w) = \rho_0(1 - w^2\hat{w}^{-2})^{1/(2k)}.$$

The B_ν can now be computed by using the following formulas derived from (1.10):

$$B_\nu = \frac{\hat{w}^2\rho_0 \int_\pi^\pi \psi_w^{(0)} e^{-i\nu\theta}\, d\theta \Big|_{w=w_0}}{2\pi i\nu\rho(w_0)w_0^{|\nu|-1}[\hat{w}^2F_{|\nu|}(w_0^2\hat{w}^{-2}) + 2w_0^2F'_{|\nu|}(w_0^2\hat{w}^{-2})]}, \qquad \nu \neq 0 \tag{1.11}$$

$$B_0 = \frac{-\hat{w}^4\rho_0(1 - w_0^2c_0^{-2})\psi_\theta^{(0)}\,|_{\theta=0}}{4\pi\rho(w_0)[2w_0^2F_0''(w_0^2\hat{w}^{-2}) + \hat{w}^2F_0'(w_0^2\hat{w}^{-2}) + w_0^2F_0(w_0^2\hat{w}^{-2})]w_0}$$

where w_0 is the value of w for the neighborhood of which we are computing the B_ν.

2. The determination of the solutions of the equations (1.7) in the transonic case. In the present section, the determination of $\Phi^{(1)}$ will be discussed; the same procedure can then be repeated for larger n.

As indicated in §1, it is convenient to replace, in the set of equations (1.5), the expression $N(\Phi)$ by $N_0(\Phi)$, see (1.7). As it will be explained in detail in the following, this permits, for small values of n, the use of tables which can be computed once and for all. On the other hand, obviously $N(\Phi)$ can be approximated by (1.7) only in a domain where the variation of $\Phi_\nu^{(n-1)}$ is comparatively small.

Accordingly, the domain B, in which $\Phi^{(0)}$ is considered, has to be divided into a number of sufficiently small domains N_μ, so that in every neighborhood N_μ the variation of $\Phi_\nu^{(0)}$ is sufficiently small. In order to determine $\Phi^{(1)}$ it will be necessary to use a different procedure, depending upon whether N_μ intersects the sonic line or not. In this section, we shall consider an N_μ which intersects the sonic line. The two remaining cases, N_μ entirely in the sub- or supersonic region, will be considered in §3 and 4.

In order to solve the equation

$$L(\Phi^{(1)}) = N_0(w, \theta) \tag{2.1}$$

in a neighborhood N_μ of the sonic line, it is convenient to replace w by a new variable

$$q = k\hat{w}^2 - (k+1)w^2. \tag{2.2}$$

The expression $L(\Phi)$ assumes then the form

$$\begin{aligned} L(\Phi) = (k+1)\,[k(k\hat{w}^2 - q)\,(\hat{w}^2 + q)]^{-1}\{&4k(k\hat{w}^2 - q)^2(\hat{w}^2 + q)\Phi_{qq} \\ &-2(k\hat{w}^2 - q)\,[k(\hat{w}^2 + q) + q(k+1)]\Phi_q + q(k+1)\Phi_{\theta\theta}\}. \end{aligned} \tag{2.3}$$

We also write

$$k(k+1)^{-1}(\hat{w}^2 + q)(k\hat{w}^2 - q)N_0 = \sum_{s=0}^{\infty} \sum_{\lambda=-(2N+1)}^{2N+1} c_{s\lambda}e^{i\lambda\theta}q^s, \tag{2.4}$$

and now apply the method of power series to find $\Phi^{(1)}$.

The result is

$$\Phi^{(1)} = A_0(\theta) + A_1(\theta)q + \sum_{n=1}^{\infty}\left[(N_{n+2})^{-1}\sum_{\lambda=-(2N+1)}^{2N+1} M_{n+2,\lambda}e^{i\lambda\theta}\right]q^{n+2} + \Phi_1^{(0)}, \tag{2.5}$$

where $\Phi_1^{(0)}$ is an arbitrary solution of (1.3), and $A_0(\theta)$ and $A_1(\theta)$ are arbitrary analytic functions of θ;

$$a_{n,\lambda} = M_{n,\lambda}/\mathrm{N}_n, \qquad \text{and} \qquad A_n(\theta) = \sum_{\lambda=-(2N+1)}^{2N+1} a_{n,\lambda}\, e^{i\lambda\theta}, \qquad \text{for } n \geq 2.$$

Here, if

$$\begin{aligned} S_n &= 4(n+3)(n+2)k^3\hat{w}^6 \\ T_n &= [4(n+1)k^3\hat{w}^4 - (8n+10)k^2\hat{w}^4](n+2) \\ U_n &= [(4n-2)k\hat{w}^2 - (8n+2)k^2\hat{w}^2](n+1) \\ V_{n,\lambda} &= [((4n-6)k-4)n - (k+1)\lambda^2]. \end{aligned} \tag{2.6}$$

then

$$\mathrm{N}_m = \prod_{i=1}^{m} S_i \tag{2.7}$$

and

(2.8) $M_{n+3,\lambda}$

$$= \begin{vmatrix} c_{n+1,\lambda} & T_n & U_n & V_{n,\lambda} & 0 & 0 & 0\cdots 0 & 0 & 0 & 0 \\ c_{n,\lambda} & S_{n-1} & T_{n-1} & U_{n-1} & V_{n-1,\lambda} & 0 & 0\cdots 0 & 0 & 0 & 0 \\ c_{n-1,\lambda} & 0 & S_{n-2} & T_{n-2} & U_{n-2} & V_{n-1,\lambda} & 0\cdots 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{3,\lambda} & 0 & 0 & 0 & \cdots & \cdots & 0\,0 & S_2 & T_2 & U_2 & V_{2,\lambda} \\ (c_{2,\lambda} - V_{1,\lambda}a_{1,\lambda}) & 0 & 0 & \cdots & \cdots & \cdots & 0\,0\,0 & S_1 & T_1 & U_1 \\ (c_{1,\lambda} - V_{0,\lambda}a_{0,\lambda} - U_0a_{1,\lambda}) & 0 & 0 & \cdots & \cdots & \cdots & 0\,0\,0 & 0 & S_0 & T_0 \\ (c_{0,\lambda} - T_{-1}a_{1,\lambda}) & 0 & 0 & 0 & \cdots & \cdots & 0\,0\,0 & 0 & 0 & S_{-1} \end{vmatrix}$$

for $n \geqq 2$;

Similar formulas hold for $M_{2,\lambda}$, $M_{3,\lambda}$, and $M_{4,\lambda}$. (See appendix)

In the computation of these constants, tables which can be prepared once and for all greatly facilitate the work.

For the $c_{s,\lambda}$ we obtain the formulas

$$c_{s,\lambda} = \begin{cases} (2i)^{-1} \displaystyle\sum_{\nu=-N}^{N} A_\nu (A_{\nu-\lambda-1} E^{(s)}_{\nu,\lambda-\nu-1} + A_{\nu-\lambda+1} E^{(s)}_{\nu,\lambda-\nu+1}) \\ \qquad\qquad \text{if } |\lambda| < 2N \\ (2i)^{-1} \displaystyle\sum_{\nu=-N}^{N} A_\nu (A_{\lambda-\nu-\delta_\lambda}) E^{(s)}_{\nu,\lambda-\nu-\delta_\lambda} \\ \qquad\qquad \text{if } \lambda = \delta_\lambda(2N) \quad \text{or} \quad \lambda = \delta_\lambda(2N+1),\ \delta_\lambda = \pm 1. \end{cases} \tag{2.9}$$

Here the $E^{(\nu)}_{\nu,\mu}$ are constants which can be determined once and for all since they are derived only from the coefficients of[4] $F_{|\nu|}$.

Therefore, *to obtain a general solution* $\Phi^{(1)}_{\mu}$ *of* (1.5) *in an* N_μ *situated in a transonic region we write*

$$\Phi^{(1)}_{\mu} = \Phi^{(0)}_{\mu} + \Phi^{(1)} \tag{2.10}$$

where $\Phi^{(0)}_{\mu} = \sum_{\nu=-\infty}^{\infty} X^{(\mu)}_{\nu}\, w^{|\nu|}\, e^{i\nu\theta} F_{|\nu|}$ *is given by* (2.9) with B_ν replaced by arbitrary constants $X^{(\mu)}_{\nu}$ which depend upon the specific solution and whose determination will be discussed in §5. On the other hand, in the solution $\Phi^{(0)}$ which does not involve any arbitrary constants depending upon the specific solution, the B_ν are completely determined, see (1.11), since we start from a known solution. $\Phi^{(1)}$ is determined as follows: Using the given B_ν, we compute $c_{\kappa,\lambda}$, see (2.9), which are substituted into (2.8) to obtain $M_{\kappa,\lambda}$. Using these values, we finally, according to (2.5) calculate $\Phi^{(1)}$.

3. Reduction of equation (1.3) to the canonical form. Determination of the fundamental solution. In the present and the next section, we shall describe *a method analogous to that described in* §2, *but where the neighborhood* N_μ *lies completely in the sub- or in the supersonic region.* A difference which now arises is that we have to operate with the characteristic variables α, β;

$$\alpha = \theta + \Lambda_1(w), \qquad \beta = \theta - \Lambda_1(w). \tag{3.1}$$

$\Lambda_1(w)$ for supersonic and subsonic cases can be found in (2.11) and (2.6) of [6], respectively. The characteristic variables α and β are in the subsonic case conjugate imaginary quantities, while in the supersonic region they become two independent real variables.

Furthermore, in order to solve the non-homogeneous equation (1.3), instead of a direct attack such as discussed in §2, *we shall use the method of so-called fundamental solutions.*

The expression (1.2a) assumes in the characteristic variables α, β the form

$$L(\Phi) = 2w^{-2}[(1 - w^2c^{-2}) - [\Lambda_1'(w)]^2w^2]\Phi_{\alpha\beta} + (\Lambda_1''(w) - [\Lambda_1'(w)]^3w)(\Phi_\alpha - \Phi_\beta). \tag{3.2}$$

Here w and c are known functions of α and β.

Besides some minor differences, the considerations in both sub- and supersonic cases proceed in almost the same manner; therefore we carry out our discussions simultaneously.

The problem of solving the system (1.5) reduces to solving for every n a non-homogeneous equation. As we mentioned before, this will be done by using so-called *fundamental solutions. We shall at first explain this method in the general case of a linear equation.*

Let a partial differential equation

$$M(\Phi) = \Phi_{\alpha\beta} + a\Phi_\alpha + b\Phi_\beta + c\Phi = 0 \tag{3.3}$$

be given, where a, b, and c are functions of α and β which are analytic in α and β

[4] For the determination of the $E^{(\nu)}_{\nu,\mu}$ see the Appendix.

in a sufficiently small domain, say D. A solution of the non-homogeneous equation

$$M(\Phi) = S(\alpha, \beta) \tag{3.4}$$

where $S(\alpha, \beta)$ is a given function which is analytic in D, can be obtained by the formula $\Phi = \iint_B W(\alpha, \beta; A, B)\, S(A,B)\, dAdB$, where

$$W(\alpha, \beta; A, B) = U(\alpha, \beta; A, B)[\log(\alpha - A) + \log(\beta - B)] + V(\alpha, \beta; A, B) \tag{3.5}$$

is called a fundamental solution of $M(\Phi) = 0$. See [5], p. 473. Here U and V can be written in the form of infinite series which converge in D.

$$U = U_0 + U_1 + U_2 + \cdots \tag{3.5}$$

where

$$U_0 = \frac{1}{2}\left[\exp\left(\int_\beta^B a(\alpha, Y)\, dY\right) + \exp\left(\int_\alpha^A b(X, \beta)\, dX\right)\right] \tag{3.7}$$

and U_{n+1} is given by

$$-U_{n+1} = \int_\alpha^A \int_\beta^B (a\partial U_n/\partial\alpha_1 + b\partial U_n/\partial\beta_1 + cU_n)\, d\alpha_1\, d\beta_1 . \tag{3.8}$$

Furthermore,

$$V = V_0 + V_1 + V_2 \cdots \tag{3.9}$$

where V_0 and V_{n+1} are given by

$$V_0 = (aU - U_\beta)(A - \alpha)^{-1} + (bU - U_\alpha)(B - \beta)^{-1}$$

$$-V_{n+1} = \int_\alpha^A \int_\beta^B (a\partial V_n/\partial\alpha_1 + b\partial V_n/\partial\beta_1 + cV_n)\, d\alpha_1 d\beta_1 . \tag{3.10}$$

A fundamental solution of the equation (3.3), which is obtained as described above, satisfies (3.3) and possesses the property that at the point (A, B), $U(\alpha, \beta; A, B) = 1$.

The method of a fundamental solution is usually applied to the equations of elliptic type, in subsonic case, in which α and β are conjugate.

In order to be able to apply it to the supersonic case, we continue the variables to complex values, and consider our functions in a four-dimensional domain B^4. Let $\gamma = \frac{1}{2}(\alpha + \beta)$ and $\delta = (2i)^{-1}(\alpha - \beta)$, where $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$. At a given point $(\gamma_1^{(0)}, \gamma_2^{(0)}, \delta_1^{(0)}, \delta_2^{(0)})$, $\Phi(\gamma_1 + i\gamma_2, \delta_1 + i\delta_2)$ satisfies simultaneously the equation:

$$(\partial^2\Phi/\partial\gamma_1^2 + \partial^2\Phi/\partial\delta_1^2)/4 + \tilde{A}(\gamma_1 + i\gamma_2^{(0)}, \delta_1 + i\delta_2^{(0)})\partial\Phi/\partial\gamma_1$$
$$+\tilde{B}(\gamma_1 + i\gamma_2^{(0)}, \delta_1 + i\delta_2^{(0)})\partial\Phi/\partial\delta_1 + \tilde{C}(\gamma_1 + i\gamma_2^{(0)}, \delta_1 + i\delta_2^{(0)})\Phi \tag{3.11}$$
$$= S(\gamma_1 + i\gamma_2^{(0)}, \delta_1 + i\delta_2^{(0)}) \text{ as a function of } \gamma_1 \text{ and } \delta_1 ,$$

and

$$(\partial^2\Phi/\partial\gamma_1^2 - \partial^2\Phi/\partial\delta_2^2)/4 + A(\gamma_1 + i\gamma_2^{(0)}, \delta_1^{(0)} + i\delta_2)\partial\Phi/\partial\gamma_1$$
$$- iB(\gamma_1 + i\gamma_2^{(0)}, \delta_1^{(0)} + i\delta_2)\partial\Phi/\partial\delta_2 + C(\gamma_1 + i\gamma_2^{(0)}, \delta_1^{(0)} + i\delta_2)\Phi \tag{3.12}$$
$$= S(\gamma_1 + i\gamma_2^{(0)}, \delta_1^{(0)} + i\delta_2) \text{ as a function of } \gamma_1 \text{ and } \delta_2 .$$

In the fundamental solution (3.5) the quantities α, β, A and B have now the following significance:

$$\begin{aligned} \alpha &= \gamma_1 + i\gamma_2^{(0)} + i(\delta_1 + i\delta_2^{(0)}) \\ \beta &= \gamma_1 + i\gamma_2^{(0)} - i(\delta_1 + i\delta_2^{(0)}) \\ A &= \Gamma + i\Delta - \delta_2^{(0)} + i\gamma_2^{(0)} \\ B &= \Gamma - i\Delta + \delta_2^{(0)} + i\gamma_2^{(0)} \end{aligned} \tag{3.13}$$

where Γ and Δ are *real* integration variables. We shall now denote the fundamental solution by $\widetilde{W}(\gamma_1, \delta_1 ; \gamma_2^{(0)}, \delta_2^{(0)}; \Gamma, \Delta) \equiv W(\alpha, \beta; A, B)$. Φ, considered as a function of γ_1, δ_1 and γ_1, δ_2 is a solution of (3.11) and (3.12) respectively. Since (3.11) is an equation of elliptic type, Φ can be written in the form

$$\Phi(\gamma_1 + i\gamma_2^{(0)}, \delta_1^{(0)} + i\delta_2)$$
$$= \iint_{\Gamma^2+\Delta^2<1} \widetilde{W}(\gamma_1, \delta_1^{(0)}, \gamma_2^{(0)}, \delta_2, \Gamma, \Delta) S(\Gamma + i\gamma_2^{(0)}, \Delta + i\delta_2)\, d\Gamma\, d\Delta \tag{3.14}$$

which expression, considered as a function of γ_1 and δ_2, represents a solution of (3.12), an equation of hyperbolic type.

REMARK: The integration has to be carried out over a circle $\Gamma^2 + \Delta^2 < 1$, in the plane $\gamma_2 = \gamma_2^{(0)}$, $\delta_2 = \delta_2^{(0)}$, see Fig. 3.1. The expression (3.14) satisfies also the equation (3.12) and therefore can be used in the supersonic case. We wish, however, to stress that in the subsonic case the integration domain K lies in the plane of the variables of the differential equation, γ_1, δ_1, while in the supersonic case K lies in the plane which is perpendicular to γ_2 and δ_1 of the equation.

4. Determination of $\Phi^{(1)}$ in N_μ belonging entirely to the sub- or supersonic region. In this section, we shall apply the formulas derived in §3 to our specific case, i.e. evaluate the formula (3.14) when (3.4) assumes the form (3.2). Since the explicit expressions for w and θ in terms of the characteristic variables α and β are rather involved, it is convenient to rewrite the formulas of §3 in terms of w and θ; this will form the main contents of the present section.

In (3.1) the function $\Lambda_1(w)$ has been introduced. In the following, we shall need the inverse function

$$w = \Omega(\Lambda_1). \tag{4.1}$$

$\Omega(\Lambda_1)$ is a function of a complex variable $\Lambda_1 = \Lambda + i\lambda$. In the hydrodynamic application, Λ_1 is real in the supersonic region, and purely imaginary in the subsonic region, i.e., $\lambda = 0$ for $M > 1$, and $\Lambda = 0$ for $M < 1$. We note, however,

that in our treatment we need the values of Ω for complex values of Λ_1. $\Omega(\Lambda_1)$ can be represented in the form of power series. In the subsonic case, the power series development of

$$T = c^{-1}[k\hat{w}^2 - w^2(1 + k)]^{1/2} \tag{4.2}$$

and[5] T^{-1} in terms of exp.(2λ) have been given in [1, pp. 49 and 75, 76]. This representation holds for $0 < T \leqq 1$. For values near $T = 0$, the convergence of the series is slow.[6] On the other hand, another series development can be obtained for this region, as follows.

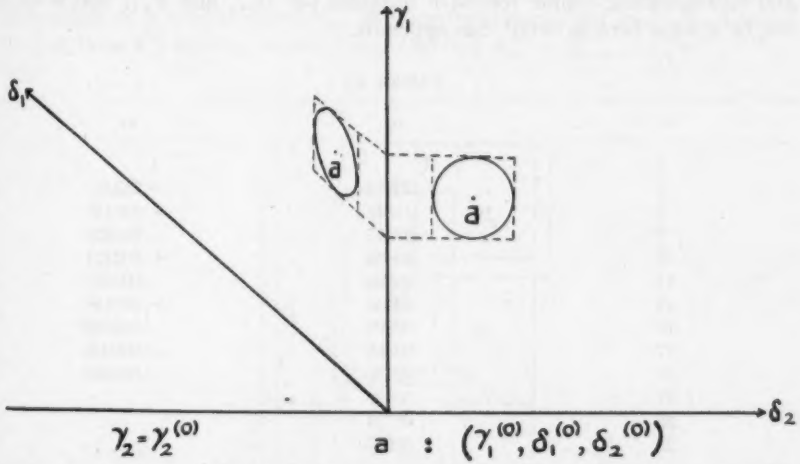

FIG. 3.1. The intersection of the four-dimensional $\gamma_1\gamma_2\delta_1\delta_2$ space with $\gamma_2 = \gamma_2^{(0)}$. The point to the left illustrates the situation in the subsonic case; the point to the right, that in the supersonic case.

In the neighborhood of the point $\lambda = 0$ for $k = 0.2(\gamma = 1.4)$, a formal computation yields the representation

$$\lambda = c_1 T + c_3 T^3 + \cdots \tag{4.3}$$

$$T = d_1\zeta + d_3\zeta^3 + \cdots \tag{4.4}$$

where $\zeta = (-3.6\lambda)^{1/2}$ and the c_n and d_n are given in the table 4.1, p. 142.

REMARK: We assume in the following that tables for the $\Omega(\Lambda_1)$ for *complex* values of the argument Λ_1 are available.

We can now substitute into the expressions (3.7) – (3.10) the variables w and θ. If we let

$$G(w) = \tfrac{1}{2}[\Lambda_1''(w)w^2 - (w\Lambda_1'(w))^2]\,(1 - w^2c^{-2} - [\Lambda_1'(w)]^2w^2)^{-1} \tag{4.5}$$

[5] We note that $T = (1 - M^2)^{1/2}$.

[6] We note that tables for $\Omega(\Lambda_1)$ are available for the supersonic region, i.e., for Λ_1 purely imaginary.

then we obtain

$$\bar{U}_0(w, w'; \theta, \Gamma; \lambda^{(0)}, \theta_2^{(0)}) = \frac{1}{2}\left\{\exp\left[2\int_{\sigma=\Omega(\frac{1}{2}(i\Delta-\lambda^{(0)}))}^{\sigma=\Omega(\frac{1}{2}(i\Delta+\theta_1-\Gamma+i\Delta-2\lambda^{(0)}))} G(\sigma)\Lambda_1'(\sigma)\,d\sigma\right]\right.$$

(4.6)

$$\left. + \exp\left[2\int_{\sigma=\Omega(\frac{1}{2}(i\Delta-\lambda^{(0)}))}^{\sigma=\Omega(\frac{1}{2}(\Gamma+i\Delta+i\Delta-\theta_1-2\lambda^{(0)}))} G(\sigma)\Lambda_1'(\sigma)\,d\sigma\right]\right\}.$$

$$\bar{V}_0 = \{i(\Lambda - \Delta)[2UG(w) - U_w(\Lambda_1'(w))^{-2}] + (\Gamma - \theta_1)U_\theta\}/[(\Gamma - \theta_1)^2 + (\Lambda - \Delta)^2] \tag{4.7}$$

and corresponding simple recursive formulas for $\bar{U}_{n+1}$ and $\bar{V}_{n+1}$ which need not be written here in detail. See appendix.

TABLE 4.1

n	c_n	d_n
1	1	1
3	.23333	−.23333
5	.11619	+.047143
7	.07477	−.010329
9	.05435	+.002721
11	.04234	−.000669
13	.03448	+.000158
15	.02896	−.000038
17	.02488	+.000009
19	.02176	−.000002
21	.01930	
23	.01731	
25	.01568	

Here and in the following, $\Lambda_1 \equiv \Lambda_1(w)$, $\Delta \equiv \Delta(w')$.

We then get, for the fundamental solution, the formula

$$\bar{W}(w, w'; \theta_1, \Gamma; \lambda^{(0)}, \theta_2^{(0)}) = \bar{U}(w, w_i'; \theta_1, \Gamma; \lambda^{(0)}, \theta_2^{(0)}) \log([(\Gamma - \theta_1)^2 + (\Delta - \Lambda)^2]) + \bar{V}(w, w'; \theta_1, \Gamma; \lambda^{(0)}, \theta_2^{(0)}). \tag{4.8}$$

In order to obtain *the solution of the non-homogeneous equation* (2.1) *where* $N_0 = N_0(\Phi^{(0)})$, *in a neighborhood* N_μ which lies *completely in a sub- or a supersonic region*, we apply the method of fundamental solutions, and, according to (3.14), *we write*

$$\Phi^{(1)} = \frac{1}{2}\iint_{\Gamma^2+[\Lambda_1(w')]^2<1} W(w, w'; \theta_1, \Gamma; \lambda^{(0)}, \theta_2^{(0)}) N_0(\Phi^{(0)}) \times G(w')\Lambda_1'(w')[\Lambda_1''(w') - w'(\Lambda_1'(w'))^3]^{-1}\,dw'\,d\Gamma + \Phi_1^{(0)} \tag{4.9}$$

where $N_0(\Phi^{(0)})$, is considered as a function of w' and Γ rather than w and θ.

Here, $\Phi_1^{(0)}$ is again an arbitrary solution.

5. Combining the elements Φ_μ into one function. By the procedure described in §2 and §3, we can, starting from a solution $\Phi^{(0)}$ of a plane flow, determine in every neighborhood N_μ a solution, say $\Phi_\mu^{(1)}$, of equation (1.5) for $n = 1$. As we stressed before, in doing this we can use tables for functions and expressions which can be computed once and for all. We remind the reader that in every N_μ the function $N(\Phi^{(0)})$ is replaced by $N_0(\Phi^{(0)})$, see (1.7), and that naturally $c_\mu = 2^{-1}(\Phi_{v,\max}^{(0)} + \Phi_{v,\min}^{(0)})$ will change from region to region. On the other hand, the tables mentioned above can be prepared for $c = 1$, and the values obtained from the tables have to be divided by c_μ.

It remains now to discuss how to combine the functions $\Phi_\mu^{(1)}$, each of which is determined in a neighborhood N_μ, into one function defined in $\sum N_\mu$. Each of the solutions $\Phi_\mu^{(1)}$ involves an arbitrary function $\Phi_\mu^{(0)} = \sum_{\nu=-\infty}^{\infty} X_\nu^{(\mu)} w^{|\nu|} e^{i\nu\theta} F_{|\nu|}$.

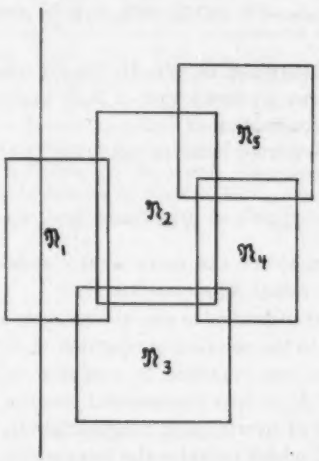

FIG. 5.1

See (1.8). (We note that in (1.8) the symbol B_ν is used instead of $X_\nu^{(\mu)}$.) We wish to indicate how to determine the constants $X_\nu^{(\mu)}$ in order that the solution obtained in this manner will approximate in the whole domain $N = \sum_\mu N_\mu$ in which $\Phi^{(0)}$ has been originally defined, one solution.

We assume here that we have only one domain N_μ, say N_1, which lies in the transonic region, and the remaining N_μ, say $N_2, N_3, N_4, \cdots$, belonging to the supersonic region. (See Fig. 5.1.)

Since we wish to determine a solution of the axially symmetric flow which differs as little as possible from the plane solution $\Phi^{(0)}$, *we determine the* $X_\nu^{(1)}$ *in such a way that*

$$\iint_{N_1} [\Phi_1^{(0)} - \Phi_1^{(1)}]^2 \, dw \, d\theta = \min. \tag{5.1}$$

The condition (5.1) permits us to determine the unknown constants $X_\nu^{(1)}$. Let

us denote the expression (2.10), into which the $X_\nu^{(1)}$, determined as above, have been substituted, by $\phi_1^{(1)}$. Let AB be a line segment in the intersection of N_1 and N_2 which is not a portion of a characteristic. The function $\phi_1^{(1)}$ and its derivative assume certain values on the line AB, and now we determine in the solution $\Phi_2^{(1)}$ the constants $X_\nu^{(2)}$ so that on the segment AB the obtained value $\phi_2^{(1)} = \Phi_2^{(0)} + \Phi_2^{(1)}$ and its derivative differs as little as possible from $\phi^{(1)}$ and its derivatives. This procedure can be continued through all N_μ so that we get a solution $\phi^{(1)}$ in the whole domain. We should like to note that if the domain involves neighborhoods N_μ belonging to the subsonic region, we can determine the constants $X_\nu^{(\mu)}$ by using the method of analytic continuation, which we shall not discuss in detail at this time.

The second approximation $\phi^{(2)}$ can be obtained, in principle, in the same manner as $\phi^{(1)}$. In this process, further tables will be required. We note, however, that presumably in most instances it will be sufficient to determine $\phi^{(1)}$.

6. Some remarks of theoretical nature. In the previous sections, we described a method of starting from a given solution of (1.3) to obtain functions which can be considered as an approximation of (1.2).

Every $\Phi_\mu^{(n)}$ can be substituted into the equation (1.2); we obtain then an inhomogeneous equation

$$L(\Phi^{(n)}) - N(\Phi^{(n)}) = M(u, v) \tag{6.1}$$

and from the size of $M(u, v)$ one can make some conclusions about the accuracy of the approximation in every N_μ.

On the other hand, *it is of interest to give at least some sufficient criteria when the* $\Phi^{(n)}$ *converge, as* $n \to \infty$, to the solution of equation (1.2). Since we are interested in analytic solutions, we can continue to complex values of the arguments.[7] We determine to every N_μ a four-dimensional domain B_μ^4 which includes N_μ, and further to each pair of overlapping neighborhoods, say N_μ and N_ν, a four-dimensional domain $M_{\mu\nu}^4$ which includes the intersection, $N_\mu \cap N_\nu$.

We proceed now to describe briefly one such criterion:

If we determined at each step the constants $X_m^{(n,\mu)}$ in $\Phi_{n,\mu}^{(0)} = \sum X^{(n,\mu)} \Phi_m^{(\mu)}(w, \theta)$ in such a way that

$$\begin{aligned} &\lambda \sum_\mu \iiiint_{B^4} |\Phi_\mu^{(n)} - \Phi_{n,\mu}^{(0)} - \Phi_\mu^{(n-1)}|^2 \, d\omega \\ &\quad + \tau \sum_{\mu,\nu} \iiiint_{M^4} |\Phi_\mu^{(n)} + \Phi_{n,\mu}^{(0)} - \Phi_\nu^{(n)} - \Phi_{n,\nu}^{(0)}|^2 \, d\omega = \text{minimum} = S_n, \end{aligned} \tag{6.2}$$

where λ and τ are two fixed numbers such that $\lambda + \tau = 1$, $\lambda > 0$, $\tau > 0$, then the sufficient condition for the convergence is that $\sum_{n=1}^{\infty} S_n^{\frac{1}{4}} < \infty$. The proof of this and other criteria can be accomplished with the help of the theory developed in [4].

[7] In the following, we employ notions and results from the theory of functions which are not contained in this paper. We refer the reader to [4].

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ON THE THEORY OF THIN ELASTIC TOROIDAL SHELLS[1]

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1. Introduction. In the present paper toroidal or ring shell problems are considered from the point of view of the small deflection theory of thin shells of revolution loaded symmetrically with respect to their axis. Solutions of such problems are found by applying methods of asymptotic integration of a differential equation involving a large parameter multiplied by a function which vanishes at certain points. As examples of the general procedure developed the particular problems of an "Omega" piping expansion joint subject to axial load and of a corrugated pipe subject to axial load or internal pressure are solved. In particular, explicit formulas are given for the maximum stresses and displacements in terms of the dimensions of the shells.

For a toroidal shell of uniform thickness and circular cross section the theory of shells of revolution reduces the problem, essentially, to that of integrating a single complex differential equation subject to boundary conditions which depend on the particular loading considered. This differential equation is of the form,

$$\frac{d^2U}{d\phi^2} + f_1(\phi, \lambda)\frac{dU}{d\phi} + [i\mu f_2(\phi, \lambda) + f_3(\phi, \lambda)]U = f_4(\phi, \lambda, \mu) \tag{1}$$

The quantities λ and μ in equation (1) are parameters defined as

$$\lambda = \frac{b}{a}, \qquad \mu = \sqrt{12(1 - \nu^2)}\,\frac{b^2}{ah}$$

where b is the radius of the circular cross section, a is the distance of the center of the cross section from the axis of revolution, h is the thickness of the shell and ν is Poisson's ratio. The parameter μ is the more significant of the two and will be large compared to unity provided h is sufficiently small compared to the other dimensions. The parameter λ is less than unity for most, if not all, applications. If λ is sufficiently small, the effect of terms in λ alone is relatively insignificant.

In the work to follow we assume that the parameter μ is large and obtain approximate solutions of eq. (1) by applying methods of asymptotic integration which differ somewhat from the methods usually applied to thin shell problems. The usual method of asymptotic integration assumes that the term involving the large parameter is large everywhere. For a toroidal shell the function $f_2(\phi, \lambda)$ in equation (1) vanishes at those points where the tangent to a meridian is perpendicular to the axis of revolution. Consequently the ordinary asymptotic integration theory is valid only for regions of a shell where $f_2(\phi, \lambda)$ is not small. For regions where $f_2(\phi, \lambda)$ is small it is possible to make a separate approximation but it is much more desirable to have a single representation for a solution

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[2] Now at Case Institute of Technology.

which is valid regardless of the size of $f_2(\phi, \lambda)$. Such a representation is found here for the solution of the homogeneous equation related to equation (1) by applying a method of asymptotic integration developed by R. E. Langer [4].[3] The results of this method coincide with the results of the usual method in regions where $f_2(\phi,\lambda)$ is not small. By using somewhat similar reasoning we also solve the more difficult problem of obtaining a formula for an *approximate particular integral* of equation (1) in a way that is independent of the precise nature of the loading terms.

The advantage of asymptotic integration methods is that the accuracy of the results so obtained increases as the parameter μ becomes larger. Power series methods, on the other hand, become less practical as μ becomes larger. However, if μ is not too large either power series methods or methods of successive approximation may be used. Considerable work has been done on toroidal shell problems using power series integration [7], [10]. In the present paper a direct check on the

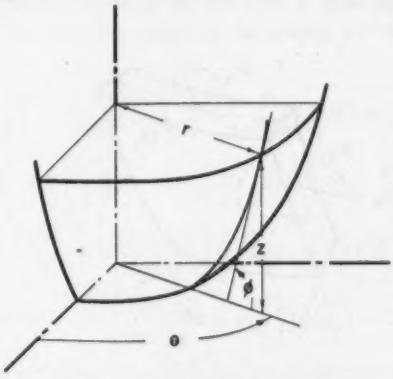

FIG. 1

accuracy of the solution obtained for the homogeneous equation is given by a comparison of the numerical results found by asymptotic integration and found by H. Wissler [10] using power series integration for two specific examples. The agreement is quite satisfactory although the value of μ used in the calculations is less than 7.

2. Formulation of the Problem. a. *Theory of shells of revolution.* In this section we give a brief summary of the usual small deflection theory of thin shells of revolution. The notation adopted is that of a recent paper by E. Reissner [6], which also contains references concerning the historical development of the subject.

Using cylindrical coordinates r, θ, z let the middle surface be represented by the parametric equations

$$r = r(\xi), \qquad z = z(\xi) \tag{2}$$

[3] Numbers in brackets correspond to the references listed in the bibliography.

and let ϕ be the angle between the r direction and the tangent to a meridian. Then,

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi$$

where

$$\alpha = \sqrt{(r')^2 + (z')^2} \tag{3}$$

and primes indicate differentiation with respect to ξ.

The stress resultants, N_ξ, N_θ and Q, and the stress couples, M_ξ and M_θ, acting on an element of a shell, are taken in accordance with figure 2. It is also convenient to define "horizontal" and "vertical" stress resultants, H and V, which are related to N_ξ and Q by the equations

$$\alpha N_\xi = r'H + z'V, \qquad \alpha Q = -z'H + r'V \tag{4a, b}$$

Denoting the components of load intensity in the radial and axial directions

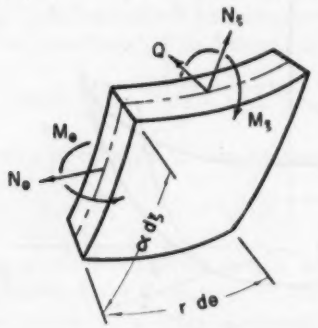

FIG. 2

by p_H and p_V, respectively, the equilibrium equations for elements of a shell may now be written in the form

$$(rV)' + r\alpha p_V = 0, \qquad (rH)' - \alpha N_\theta + r\alpha p_H = 0 \tag{5a, b, c}$$
$$(rM_\xi)' - r'M_\theta - r\alpha Q = 0$$

Assuming that the behavior of the material is isotropic, the stress resultants and couples are related to direct and bending strains by the equations

$$C\epsilon_{\xi M} = N_\xi - \nu N_\theta, \qquad C\epsilon_{\theta M} = N_\theta - \nu N_\xi \tag{6a, b}$$

$$M_\xi = D(\kappa_\xi + \nu\kappa_\theta), \qquad M_\theta = D(\kappa_\theta + \nu\kappa_\xi) \tag{7a, b}$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \qquad C = Eh,$$

E is the modulus of elasticity, ν is Poisson's ratio, and h is the thickness of the shell.

Let u and w be the components of displacement in the r and z directions, respectively, and let the rotation β be the negative of the change, due to deformation, in the tangent angle ϕ. There is no displacement in the circumferential or θ direction as the loads are assumed to be axially symmetric.

For small displacements the strains in equations (6) and (7) are expressed in terms of u and β as follows

$$\epsilon_{\xi M} = \frac{u' - z'\beta}{r'}, \qquad \epsilon_{\theta M} = \frac{u}{r} \tag{8a, b}$$

$$\kappa_\xi = \frac{\beta'}{\alpha}, \qquad \kappa_\theta = \frac{r'\beta}{r\alpha} \tag{9a, b}$$

Equations (8) immediately imply the compatibility relation

$$(r\epsilon_{\theta M})' - r'\epsilon_{\xi M} = z'\beta \tag{10}$$

The next and most important step in the theory is to reduce the system of equations (4) to (10) to the two following equations for the two functions β and rH.

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right]\beta + \frac{z'}{(rD/\alpha)}(rH) = \frac{r'}{(rD/\alpha)}(rV) \tag{11a}$$

$$\begin{aligned}(rH)'' + \frac{(r/C\alpha)'}{(r/C\alpha)}(rH)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/C\alpha)'}{(r/C\alpha)}\right](rH) - \frac{z'}{(r/C\alpha)}\beta \\ = \left[\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha)'}{(r/C\alpha)}\right](rV) + \nu\frac{z'}{r}(rV)' \\ - \left[\frac{(r/C\alpha)'}{(r/C\alpha)} + \nu\frac{r'}{r}\right](r\alpha p_B) - (r\alpha p_B)'\end{aligned} \tag{11b}$$

One obtains equation (11a) by combining equations (4b), (5c), (7) and (9), while equation (11b) results from the combination of equations (4a), (5b), (6) and (10).

For a particular loading the quantity rV is given by equation (5a). Equations (11) are then solved for β and rH. The stress resultants are expressed in terms of these solutions and the components of load intensity by equations (4) and (5b). The stress couples are determined from equations (7) and (9). The displacement u results from either of equations (8) in conjunction with equations (6). The displacement w is obtained from the equation

$$w = \int (z'\epsilon_{\xi M} - r'\beta)\, d\xi \tag{12}$$

b. *Equations for a toroidal shell of uniform thickness and circular cross section.* Taking the parameter ξ in equations (2) as the tangent angle ϕ (which is also the same as the central angle of the circular cross section—see Fig. 3) the parametric equations of the middle surface of a toroidal shell are

$$r = a + b\sin\phi, \qquad z = -b\cos\phi \tag{13}$$

where b is the radius of the circular cross section and a is the distance of the center of the cross section from the axis of revolution. From equation (3) we have $\alpha = b$. The form equations (11) assume may now be determined, but before this is done it is convenient to define the parameters

$$\lambda = \frac{b}{a}, \qquad \mu = \frac{mb^2}{ah}, \qquad m = \sqrt{12(1-\nu^2)} \tag{14a, b, c}$$

and to introduce the following *dimensionless* functions,

$$\Psi = m\frac{rH}{Eh^2}, \qquad \Omega = m\frac{rV}{Eh^2}$$
$$P_H = m\frac{rbp_H}{Eh^2}, \qquad P_V = m\frac{rbp_V}{Eh^2} \tag{15a, b, c, d}$$

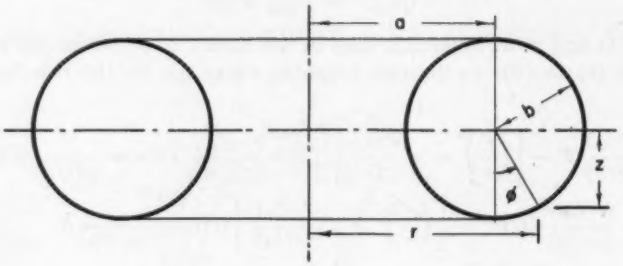

FIG. 3

Equations (11) may then be written as follows

$$L(\beta) - \nu\lambda\beta + \mu\Psi = F \tag{16}$$

$$L(\Psi) + \nu\lambda\Psi - \mu\beta = G \tag{17}$$

where

$$L(\ ) = \frac{1+\lambda\sin\phi}{\sin\phi}\left[(\ )'' + \frac{\lambda\cos\phi}{1+\lambda\sin\phi}(\ )' - \left(\frac{\lambda\cos\phi}{1+\lambda\sin\phi}\right)^2(\ )\right] \tag{18}$$

$$F = \mu\Omega\cot\phi \tag{19}$$

$$G = \left(\nu\lambda\cot\phi + \frac{\lambda^2\cos\phi}{1+\lambda\sin\phi}\right)\Omega - \nu\lambda P_V - (1+\nu)\lambda P_H\cot\phi - \frac{1+\lambda\sin\phi}{\sin\phi}P_H' \tag{20}$$

By introducing the complex function

$$U = \beta + ik\Psi \tag{21}$$

where

$$k = \sqrt{1+(\nu\lambda/\mu)^2} + i(\nu\lambda/\mu) \tag{22}$$

equations (16) and (17) may be combined into one complex equation

$$L(U) - i\sqrt{\mu^2 - (\nu\lambda)^2}\,U = F + iG. \tag{23}$$

By setting $W = \sqrt{1 + \lambda \sin \phi}\; U$ the first derivative term is eliminated and equation (23) becomes

$$W'' - \left[\left(\frac{\lambda}{2} + i\sqrt{\mu^2 - (\nu\lambda)^2}\right)\frac{\sin\phi}{1 + \lambda\sin\phi} + \frac{3}{4}\left(\frac{\lambda\cos\phi}{1 + \lambda\sin\phi}\right)^2\right] W = (F + iG)\frac{\sin\phi}{\sqrt{1 + \lambda\sin\phi}} \tag{24}$$

At this point we recall that the present theory applies only to shells which are *thin*, in the sense that the ratio $h/b \ll 1$, and that terms of the order h/b are neglected, compared with unity, in the derivation of this theory. From equations (14) we have

$$\frac{\lambda}{\mu} = \frac{h}{mb} = O\left(\frac{h}{b}\right) \tag{25}$$

It follows, then, that λ is always *small* compared with μ, and that we are only consistent in setting

$$(\lambda/2) + i\sqrt{\mu^2 - (\nu\lambda)^2} \approx i\mu, \qquad k \approx 1 \tag{26}$$

With these approximations equation (23) takes the final form

$$W'' - \left[i\mu\frac{\sin\phi}{1 + \lambda\sin\phi} + \frac{3}{4}\left(\frac{\lambda\cos\phi}{1 + \lambda\sin\phi}\right)^2\right] W = (F + iG)\frac{\sin\phi}{\sqrt{1 + \lambda\sin\phi}} \tag{27}$$

where now

$$W = \sqrt{1 + \lambda\sin\phi}\;(\beta + i\Psi) \tag{28}$$

or

$$\beta = (1 + \lambda\sin\phi)^{-(1/2)}W_r, \qquad \Psi = (1 + \lambda\sin\phi)^{-(1/2)}W_i \tag{29}$$

(Subscripts r and i will always denote the real and imaginary parts of a complex function.)

From equations (4), (5), (7) and (9) we have the following formulas for the stress resultants and couples in terms of β, Ψ and the loading functions, P_H and P_V:

$$\Omega = -\int P_V\, d\phi \tag{30}$$

$$N_\phi = \frac{Eh^2}{ma}\left(\frac{\Psi\cos\phi + \Omega\sin\phi}{1 + \lambda\sin\phi}\right) \tag{31}$$

$$Q = \frac{Eh^2}{ma}\left(\frac{-\Psi\sin\phi + \Omega\cos\phi}{1 + \lambda\sin\phi}\right) \tag{32}$$

$$N_\theta = \frac{Eh^2}{mb}(\Psi' + P_H) \tag{33}$$

$$M_\phi = \frac{Eh^3}{m^2 b}\left(\beta' + \frac{\nu\lambda\cos\phi}{1+\lambda\sin\phi}\beta\right) \tag{34}$$

$$M_\theta = \frac{Eh^3}{m^2 b}\left(\nu\beta' + \frac{\lambda\cos\phi}{1+\lambda\sin\phi}\beta\right) \tag{35}$$

In order to determine the maximum values of the principal stresses, σ_ϕ and σ_θ, we write

$$\begin{aligned} |\,\sigma_{\phi,\max}\,| &= |\,\sigma_{\phi D}\,| + |\,\sigma_{\phi B}\,| \\ |\,\sigma_{\theta,\max}\,| &= |\,\sigma_{\theta D}\,| + |\,\sigma_{\theta B}\,| \end{aligned} \tag{36}$$

where the subscripts D and B refer to the direct and bending stresses defined by the equations

$$\begin{aligned} \sigma_{\phi D} &= \frac{N_\phi}{h}, \qquad \sigma_{\theta D} = \frac{N_\theta}{h} \\ \sigma_{\phi B} &= \frac{6M_\phi}{h^2}, \qquad \sigma_{\theta B} = \frac{6M_\theta}{h^2} \end{aligned} \tag{37}$$

Combining equations (8b), (6b), (31) and (33) we obtain the following formula for the horizontal displacement,

$$u = \frac{a}{Eh}(1+\lambda\sin\phi)(N_\theta - \nu N_\phi) \tag{38}$$

An alternate formula is obtained by using equation (8a); namely,

$$u = b\int(\beta\sin\phi + \epsilon_{\phi M}\cos\phi)\,d\phi \tag{39}$$

where, from equations (6a), (31) and (33),

$$\epsilon_{\phi M} = \frac{h}{mb}\left[\lambda\frac{\Psi\cos\phi + \Omega\sin\phi}{1+\lambda\sin\phi} - \nu(\Psi' + P_H)\right] \tag{40}$$

From equation (12) the displacement w is given by

$$w = -b\int(\beta\cos\phi - \epsilon_{\phi M}\sin\phi)\,d\phi \tag{41}$$

3. Asymptotic Integration of the Homogeneous Equation. Let us first consider the homogeneous equation

$$W'' - \left[\frac{i\mu\sin\phi}{1+\lambda\sin\phi} + \frac{3}{4}\left(\frac{\lambda\cos\phi}{1+\lambda\sin\phi}\right)^2\right]W = 0 \tag{42}$$

obtained by setting the right hand side of eq. (27) equal to zero. The solution of eq. (42) is all that is required for problems dealing with a portion of a toroidal

shell subjected only to radial edge forces or edge couples, since then p_R, p_T and V are all zero and the right hand side of equation (27) vanishes identically. Restricting attention to problems for which $\lambda < 1$ (as is the case for most, if not all, practical applications) and $\mu \gg 1$, we shall now determine approximate solutions of eq. (42) asymptotic in μ to exact solutions. This is done by applying a method developed by R. E. Langer [4] for treating general problems of this type. For the range where $\mu \sin \phi$ is large the results are the same as those given by the usual method of asymptotic integration which H. Reissner and O. Blumenthal first introduced into shell theory [6].

The briefest way to arrive at the asymptotic solution of equation (42) is as follows. Let a function ω be defined as

$$\omega(\phi, \lambda) = \int_0^\phi \left(\frac{\sin \phi}{1 + \lambda \sin \phi}\right)^{1/2} d\phi \tag{43}$$
$$= \tfrac{2}{3} \sin^{3/2}\phi \left[1 - \tfrac{3}{10}\lambda \sin \phi + \tfrac{3}{14}(1 + \tfrac{3}{4}\lambda^2) \sin^2 \phi + \cdots\right]$$

The following functions x and Q are then defined in terms of ω by the relations

$$x(\phi, \lambda, \mu) = \mu^{1/3}(\tfrac{3}{2}\omega)^{2/3} \tag{44}$$

$$Q(\phi, \lambda) = \frac{(\frac{3}{2}\omega)^{1/6}}{(\omega')^{1/2}} \tag{45}$$

It should be noted that ω and x are zero when ϕ is zero. The function Q is indeterminate at the origin but may be defined there as $\lim_{\phi \to 0} Q(\phi, \lambda) = 1$. The useful relation $dx/d\phi = \mu^{1/3}/Q^2$ also holds.

Next consider the function

$$Y(\phi) = Q(\phi)h(ix) \tag{46}$$

where $h(z)$ is any solution of the equation

$$d^2h/dz^2 + z\,h = 0 \tag{47}$$

By direct substitution it is easily verified that the function Y satisfies the equation

$$\frac{d^2Y}{d\phi^2} - \left[\frac{i\mu \sin \phi}{1 + \lambda \sin \phi} + \frac{Q''}{Q}\right] Y = 0 \tag{48}$$

where

$$\frac{Q''}{Q} = (1 + \lambda \sin \phi)^{-2}\left[\tfrac{5}{16} \csc^2 \phi + \frac{\lambda}{2} \csc \phi - \frac{1}{16}\right.$$
$$\left. - \frac{\lambda}{4} \sin \phi - \frac{5}{36} \frac{\sin \phi(1 + \lambda \sin \phi)}{\omega}\right]$$
$$= \frac{1}{14} - \frac{6}{35}\lambda^2 + O(\lambda \sin \phi, \sin^2 \phi)$$

Eq. (48) resembles eq. (42) to the extent of the term in μ and has the added advantage that it is explicitly solvable since eq. (47) admits as solutions functions of the form $z^{1/2}H_{1/3}(\frac{2}{3}z^{3/2})$, where $H_{1/3}$ stands for any Bessel function of order 1/3. Because eqs. (42) and (48) differ only by terms independent of μ it is reasonable to assume that solutions of (48) are asymptotic in μ to solutions of (42) provided the terms independent of μ are always of the order of unity. It follows from the general results obtained by Langer that this is actually so if ϕ is restricted to an interval such that $|\phi| < \phi_1 < \pi$. (This restriction is necessary since the term Q''/Q becomes infinite as $\phi \to \pm\pi$, whereas the coefficient of W in eq. (42) remains finite.)

It has been found convenient to choose as specific solutions of eq. (47) the functions,

$$\begin{aligned} h_1(z) &= (\tfrac{2}{3}z^{3/2})^{1/3}H^{(1)}_{1/3}(\tfrac{2}{3}z^{3/2}) \\ h_2(z) &= (\tfrac{2}{3}z^{3/2})^{1/3}H^{(2)}_{1/3}(\tfrac{2}{3}z^{3/2}) \end{aligned} \tag{49}$$

where $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ stand for Hankel functions of order 1/3 of the first and second kind, respectively. The reason for this choice is that the particular functions $h_1(z)$ and $h_2(z)$ and their derivatives have been tabulated for complex arguments [3]. When $|z|$ is large h_1 and h_2 may be represented by appropriate asymptotic forms. For purely imaginary arguments we have

$$h_j(ix) \sim \gamma(ix)^{-1/4}e^{(-1)^j[(2/3)i^{1/2}x^{3/2}+5\pi i/12]}, \qquad j = 1, 2 \tag{50}$$

where

$$\gamma = 2^{1/3}3^{1/6}\pi^{-1/2} = 0.85367.$$

According to the work of Langer two independent solutions, $W_1(\phi)$ and $W_2(\phi)$, of eq. (42) may now be represented by the formulas

$$\begin{aligned} W_j(\phi) &= Qh_j(ix) + \frac{O(1)}{\mu^{1/2}}, && |x| \leq N \\ &= Qh_j(ix) + x^{-(1/4)}\, e^{(-1)^j(2/3)i^{1/2}x^{3/2}}\frac{O(1)}{\mu^{1/2}}, && |x| > N \end{aligned} \tag{51}$$

where $O(1)$ stands for a function bounded in the range indicated and where the statement $|x| > N$ simply implies that $|x|$ is large enough for the asymptotic forms (50) to be used. It is in the range $|x| > N$ that the formulas (51) correspond to the results of the usual method of asymptotic integration.

We now take as an approximate general solution of eq. (42)

$$W = Q[(A + Bi)h_1(ix) + (C + Di)h_2(ix)] \tag{52}$$

where A, B, C and D are real arbitrary constants. Since $dx/d\phi = \mu^{1/3}/Q^2$, the derivative of (52) may be written as

$$W' = \frac{iu^{1/3}}{Q}[(A + Bi)h_1'(ix) + (C + Di)h_2'(ix)] + \frac{Q'}{Q}W \tag{53}$$

Recalling eqs. (29), eqs. (52) and (53) imply the following approximate formulas for β, Ψ and their derivatives.

$$\beta = (1 + \lambda \sin \phi)^{-(1/2)} Q[Ah_{1,r}(ix) - Bh_{1,i}(ix) + Ch_{2,r}(ix) - Dh_{2,i}(ix)] \quad (54a)$$

$$\Psi = (1 + \lambda \sin \phi)^{-(1/2)} Q[Bh_{1,r}(ix) + Ah_{1,i}(ix) + Dh_{2,r}(ix) + Ch_{2,i}(ix)] \quad (54b)$$

$$\beta' = \frac{-\mu^{1/3}}{Q(1 + \lambda \sin \phi)^{1/2}} [Bh'_{1,r}(ix) + Ah'_{1,i}(ix) + Dh'_{2,r}(ix) + Ch'_{2,i}(ix)] + R\beta \quad (54c)$$

$$\Psi' = \frac{\mu^{1/3}}{Q(1 + \lambda \sin \phi)^{1/2}} [Ah'_{1,r}(ix) - Bh'_{1,i}(ix) + Ch'_{2,r}(ix) - Dh'_{2,i}(ix)] + R\Psi \quad (54d)$$

where subscripts r and i denote real and imaginary parts of complex functions and where

$$\begin{aligned} R(\phi, \lambda) &= \frac{Q'}{Q} - \frac{1}{2} \frac{\lambda \cos \phi}{1 + \lambda \sin \phi} \\ &= \frac{\sin^{1/2} \phi}{6\omega(1 + \lambda \sin \phi)^{1/2}} - \frac{1}{4} \frac{\cot \phi + 2\lambda \cos \phi}{1 + \lambda \sin \phi} \\ &= -\frac{3}{10} \lambda + \left(\frac{1}{14} + \frac{101}{350} \lambda^2\right) \sin \phi + O(\sin^2 \phi) \end{aligned}$$

In summary, formulas (54) furnish valid approximations for β, Ψ and their derivatives if $\mu \gg 1$, $\lambda < 1$, and $|\phi| < \phi_1 < \pi$. When ϕ is positive, $\omega(\phi, \lambda)$, $x(\phi, \lambda, \mu)$ and $Q(\phi, \lambda)$ are all real and positive. When ϕ is negative, $\omega(\phi, \lambda)$ and $\omega'(\phi, \lambda)$ are purely imaginary with arguments $3\pi/2$ and $\pi/2$, respectively. The function $Q(\phi, \lambda)$ is again real and positive whereas $x(\phi, \lambda, \mu)$ is real and negative.

When λ is small enough to be neglected, ω is given by the formula

$$\begin{aligned} \omega(\phi, 0) &= \int_0^{\phi} \sin^{1/2} \phi \, d\phi = \sqrt{2} \int_{\eta}^{\pi/2} \frac{\cos^2 \eta \, d\eta}{\sqrt{1 - \tfrac{1}{2} \sin^2 \eta}} \\ &= 2\sqrt{2}\left[E\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) - E\left(\eta, \frac{1}{\sqrt{2}}\right)\right] - \sqrt{2}\left[F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) - F\left(\eta, \frac{1}{\sqrt{2}}\right)\right] \end{aligned}$$

where $\cos^2 \eta = \sin \phi$ and where F and E represent elliptic integrals of the first and second kind, respectively. For $\lambda = 0, 0.1, 0.2, 0.3, 0.4, 0.5$, the function $\omega(\phi, \lambda)$ has been calculated by numerical integration. The results are listed in Table 1.

So far we have determined that the two independent solutions of eq. (48),

$$Y_j(\phi, \lambda, \mu) = Q(\phi, \lambda) h_j[ix(\phi, \lambda, \mu)], \qquad j = 1, 2$$

are approximate solutions of the basic differential eq. (42), in the sense of eqs. (51), only for the range $|\phi| \leq \phi_1 < \pi$. Let us now consider the problem of

TABLE I

Values of $\omega(\phi, \lambda) = \int_0^{\phi} \left(\frac{\sin \phi}{1 + \lambda \sin \phi} \right)^{1/2} d\phi$

ϕ°	$\lambda = 0$	$\lambda = .1$	$\lambda = .2$	$\lambda = .3$	$\lambda = .4$	$\lambda = .5$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0044	0.0044	0.0043	0.0043	0.0043	0.0043
4	0.0123	0.0123	0.0123	0.0122	0.0122	0.0122
6	0.0226	0.0225	0.0224	0.0224	0.0223	0.0222
8	0.0348	0.0346	0.0345	0.0343	0.0342	0.0341
10	0.0486	0.0483	0.0481	0.0478	0.0476	0.0474
12	0.0638	0.0634	0.0630	0.0626	0.0623	0.0619
14	0.0804	0.0798	0.0792	0.0787	0.0781	0.0776
16	0.0981	0.0973	0.0965	0.0958	0.0950	0.0943
18	0.1170	0.1159	0.1149	0.1139	0.1129	0.1119
20	0.1369	0.1355	0.1342	0.1329	0.1316	0.1304
22	0.1578	0.1560	0.1543	0.1527	0.1512	0.1496
24	0.1796	0.1774	0.1754	0.1734	0.1715	0.1696
26	0.2023	0.1997	0.1971	0.1947	0.1924	0.1902
28	0.2258	0.2227	0.2197	0.2168	0.2141	0.2114
30	0.2501	0.2464	0.2429	0.2395	0.2363	0.2332
32	0.2751	0.2708	0.2667	0.2628	0.2591	0.2556
34	0.3009	0.2959	0.2912	0.2867	0.2825	0.2784
36	0.3273	0.3216	0.3162	0.3111	0.3063	0.3017
38	0.3544	0.3479	0.3418	0.3360	0.3306	0.3255
40	0.3821	0.3748	0.3679	0.3614	0.3554	0.3496
42	0.4104	0.4022	0.3945	0.3873	0.3805	0.3742
44	0.4392	0.4300	0.4215	0.4135	0.4061	0.3990
46	0.4686	0.4584	0.4490	0.4402	0.4320	0.4243
48	0.4984	0.4872	0.4769	0.4672	0.4582	0.4498
50	0.5287	0.5165	0.5051	0.4946	0.4848	0.4756
52	0.5595	0.5461	0.5338	0.5223	0.5117	0.5018
54	0.5907	0.5761	0.5627	0.5503	0.5389	0.5281
56	0.6223	0.6065	0.5920	0.5786	0.5663	0.5547
58	0.6542	0.6372	0.6216	0.6072	0.5939	0.5816
60	0.6866	0.6682	0.6514	0.6360	0.6218	0.6086
62	0.7192	0.6995	0.6816	0.6651	0.6499	0.6358
64	0.7522	0.7311	0.7119	0.6944	0.6782	0.6632
66	0.7854	0.7629	0.7425	0.7238	0.7067	0.6908
68	0.8189	0.7949	0.7733	0.7535	0.7353	0.7185
70	0.8526	0.8272	0.8042	0.7833	0.7641	0.7464
72	0.8866	0.8596	0.8353	0.8132	0.7930	0.7743
74	0.9207	0.8922	0.8666	0.8433	0.8220	0.8024
76	0.9550	0.9250	0.8980	0.8735	0.8512	0.8306
78	0.9894	0.9579	0.9295	0.9038	0.8804	0.8588
80	1.0240	0.9909	0.9612	0.9342	0.9097	0.8872
82	1.0587	1.0240	0.9929	0.9647	0.9391	0.9155
84	1.0935	1.0571	1.0246	0.9952	0.9685	0.9440
86	1.1284	1.0904	1.0564	1.0258	0.9979	0.9724
88	1.1632	1.1236	1.0883	1.0564	1.0274	1.0009
90	1.1981	1.1569	1.1202	1.0870	1.0569	1.0294

determining approximate formulas which represent solutions of eq. (42) over an interval that includes the point $\phi = \pi$ as well as $\phi = 0$.

It is first observed that eq. (48) remains unchanged if the variable ϕ is replaced by the variable $\pi - \phi$. Consequently the functions $Y_1(\pi - \phi)$ and $Y_2(\pi - \phi)$ are also solutions of eq. (48) and, in turn, represent approximate solutions of eq. (42) for the range $|\pi - \phi| \leq \phi_1 < \pi$ or $\pi - \phi_1 \leq \phi \leq \pi + \phi_1$. We now have two fundamental sets of solutions of eq. (48). Consequently, it must be possible to express each solution of one set as a linear combination of the solutions in the other set; that is, we must have

$$\begin{aligned} Y_j(\phi) &= c_{1j}Y_1(\pi - \phi) + c_{2j}Y_2(\pi - \phi) \\ Y_j'(\phi) &= -c_{1j}Y_1'(\pi - \phi) - c_{2j}Y_2'(\pi - \phi) \end{aligned} \qquad j = 1, 2$$

for all values of ϕ in the interval $(0, \pi)$ except 0 and π, where either one or the other set of solutions is undefined, since $Q(\phi)$ becomes infinite at $\phi = \pi$. The constants c_{ij} may be determined from these equations in terms of the values of the functions at any interior point of the interval $(0, \pi)$. In particular, for $\phi = \pi/2$ one obtains

$$c_{11} = -c_{22} = \frac{1}{\mathbf{W}}\left[Y_1\left(\frac{\pi}{2}\right)Y_2'\left(\frac{\pi}{2}\right) + Y_1'\left(\frac{\pi}{2}\right)Y_2\left(\frac{\pi}{2}\right)\right],$$

$$c_{21} = -\frac{2}{\mathbf{W}}Y_1\left(\frac{\pi}{2}\right)Y_1'\left(\frac{\pi}{2}\right), \qquad c_{12} = \frac{2}{W}Y_2\left(\frac{\pi}{2}\right)Y_2'\left(\frac{\pi}{2}\right)$$

where $\mathbf{W}(Y_1, Y_2) = (\frac{3}{2})^{1/3}(4/\pi)\mu^{1/3}$ is the *Wronskian* of $Y_1(\phi)$ and $Y_2(\phi)$. (This is easily determined using the fact that $\mathbf{W}(h_1, h_2) = -(\frac{3}{2})^{1/3}(4/\pi)i$.)

At $\phi = \pi/2$ the asymptotic forms for $Y_j(\phi)$, found by using expressions (50) together with eq. (45), give us

$$Y_j\left(\frac{\pi}{2}\right) \sim \frac{2^{1/3}3^{1/6}}{\pi^{1/2}\mu^{1/12}}\, e^{(-1)^j[(i\mu)^{1/2}\omega(\pi/2,\lambda)+(5\pi/12)i]-(\pi/8)i}$$

$$Y_j'\left(\frac{\pi}{2}\right) \sim (-1)^j\,\frac{2^{1/3}3^{1/6}}{\pi^{1/2}}\,\mu^{5/12}\, e^{(-1)^j[(i\mu)^{1/2}\omega(\pi/2,\lambda)+(5\pi/12)i]+(\pi/8)i}$$

If these values are substituted in the equations for the c_{ij} there results

$$\begin{aligned} c_{11} &= c_{22} = 0 \\ c_{21} &= e^{-2(i\mu)^{1/2}\omega(\pi/2,\lambda)-(5\pi/6)i} \\ c_{12} &= e^{2(i\mu)^{1/2}\omega(\pi/2,\lambda)+(5\pi/6)i} \end{aligned}$$

Using these values of the constants c_{ij} we may finally write that for sufficiently large values of μ two independent solutions of eq. (42) are approximately represented by the formulas

$$W_1(\phi) = \begin{cases} Q(\phi)h_1[ix(\phi)], & -(\pi/2) \leq \phi \leq \pi/2 \\ c_{21}Q(\pi - \phi)h_2[ix(\pi - \phi)], & \pi/2 \leq \phi \leq 3\pi/2 \end{cases} \tag{55a}$$

$$W_2(\phi) = \begin{cases} Q(\phi)h_2[ix(\phi)], & -(\pi/2) \leq \phi \leq \pi/2 \\ c_{12}Q(\pi - \phi)h_1[ix(\pi - \phi)], & \pi/2 \leq \phi \leq 3\pi/2 \end{cases} \tag{55b}$$

Since eq. (42) does not possess periodic solutions except for particular values of μ, $W_j(-(\pi/2))$ is, in general, not equal to $W_j(3\pi/2)$; that is, these solutions do not apply to problems involving closed shells.

For a shell which extends for some interval about $\phi = -(\pi/2)$ we obtain in a similar way, by matching solutions at $-(\pi/2)$ instead of $\pi/2$, the formulas

$$W_1(\phi) = \begin{cases} d_{21}Q(-\pi-\phi)h_2[ix(-\pi-\phi)], & -(3\pi/2) \le \phi \le -(\pi/2) \\ Q(\phi)h_1[ix(\phi)], & -(\pi/2) \le \phi \le \pi/2 \end{cases} \tag{56a}$$

$$W_2(\phi) = \begin{cases} d_{12}Q(-\pi-\phi)h_1[ix(-\pi-\phi)], & -(3\pi/2) \le \phi \le -(\pi/2) \\ Q(\phi)h_2[ix(\phi)], & -(\pi/2) \le \phi \le \pi/2 \end{cases} \tag{56b}$$

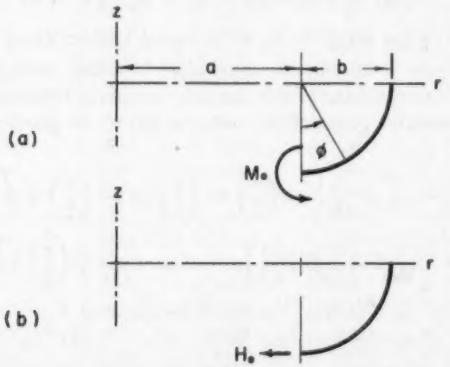

FIG. 4

where

$$d_{21} = e^{-2(i\mu)^{1/2}\omega(-(\pi/2),\lambda)-(5\pi/6)i}$$

$$d_{12} = e^{2(i\mu)^{1/2}\omega(-(\pi/2),\lambda)+(5\pi/6)i}$$

4. Examples involving the solution of the homogeneous equation only. As an explicit check on the accuracy of formulas (54) the following two examples, solved by H. Wissler [10] using power series, were worked out numerically by the author.

Example 1. Consider the portion of a toroidal shell with a cross section as shown in Fig. 4a and suppose that the shell is subjected only to a bending moment M_0 applied uniformly around the edge $\phi = 0$. Then p_H, p_V, $V = 0$, and the boundary conditions at the two edges are

$$\begin{aligned} &\text{at } \phi = 0: && M_\phi = M_0, && N_\phi = 0 \\ &\text{at } \phi = \pi/2: && M_\phi = 0, && Q = 0 \end{aligned}$$

In terms of β and Ψ (see eqs. (31) and (34)) these conditions become

$$\begin{aligned} &\phi = 0: && \beta' + \nu\lambda\beta = (m^2b/Eh^3)M_0, && \Psi = 0 \\ &\phi = \pi/2: && \beta' = 0, && \Psi = 0 \end{aligned} \tag{57}$$

Substituting from formulas (54) into eqs. (57) four algebraic equations are obtained for the four arbitrary constants A, B, C and D. Once these constants have been determined the various stress resultants and couples are found by using eqs. (31) to (35). The direct and bending stresses are then given by eqs. (37).

Example 2. Suppose that a radial force H_0 is applied uniformly around the edge $\phi = 0$ instead of a moment. The boundary conditions then become

$$\text{at } \phi = 0: \quad M_\phi = 0, \qquad N_\phi = H_0$$

$$\text{at } \phi = \pi/2: M_\phi = 0, \qquad Q = 0$$

or, in terms of β and Ψ,

$$\begin{aligned} \phi = 0: &\quad \beta' + \nu\lambda\beta = 0, \qquad \Psi = (ma/Eh^2)H_0 \\ \phi = \pi/2: &\quad \beta' = 0, \qquad \Psi = 0 \end{aligned} \tag{58}$$

and one proceeds as before.

In the numerical calculations for these examples the values of the parameters were taken as

$$\lambda = 0.1, \qquad b/h = 20, \qquad \nu = 0.3; \quad \text{hence,} \quad \mu = 6.61.$$

The direct and bending stresses given by eqs. (37) and the maximum shearing stress $\tau_0 = 3Q/2h$ are plotted in figures 5 and 6, where $\sigma_{M_0} = 6M_0/h^2$ and $\sigma_{H_0} = H_0/h$. Wissler's results are indicated by the broken curves. For example 1 there is a disagreement of about 4 percent in the maximum value of the principal direct stress $\sigma_{\theta D}$. For example 2 the maximum values of $\sigma_{\theta D}$ differ by less than one per cent, the maximum values of $\sigma_{\phi B}$ by about 3 per cent. It is rather surprising that formulas (54) proved to be as accurate as this since the value of μ used is not particularly large. This is probably explained by the fact that the difference between the coefficients in eqs. (42) and (48) is numerically small as well as being independent of μ.

If certain further approximations are made simpler formulas than eqs. (54) may be obtained which make it possible to solve some problems explicitly without specifying the values of the parameters. In particular, if loads are applied only at an edge, as in examples 1 and 2, one expects an "edge effect" if the shell is very thin; that is, the stresses are largest at or near the loaded edge and decrease rapidly as the distance from the edge increases. Now, in the general solution (52) of equation (42) the magnitude of one of the functions h_1 and h_2 will decrease exponentially, that of the other increase exponentially, as the distance from an edge increases. If an edge effect is expected then only that function which decreases in magnitude away from a loaded edge need be retained.

Let us consider examples 1 and 2 from this present point of view. We retain in eq. (52) only the function $h_1(ix)$ which has the required behavior, according to eqs. (50). Since we are primarily interested in the stresses for small values of ϕ,

we may take $\sin \phi \approx \phi$, $\cos \phi \approx 1$. Terms in $\lambda \sin \phi$ and $\nu\lambda$ are also neglected since these terms will ordinarily have only a secondary effect on the solutions. With these approximations it follows that $x = \mu^{1/3}\phi$ and $Q \equiv 1$. Hence, formulas

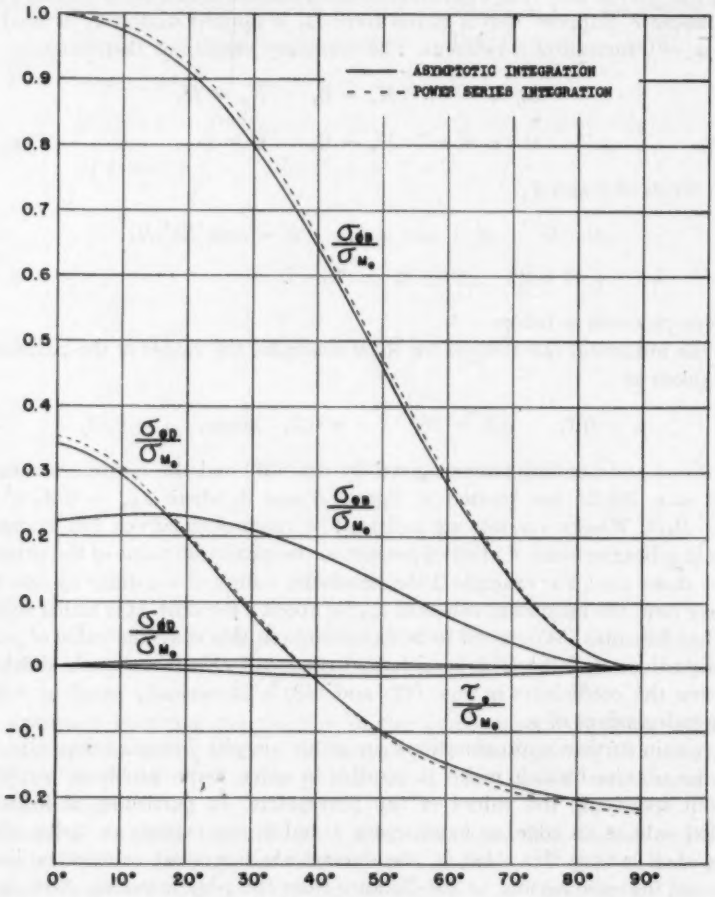

FIG. 5

(54) become simply

$$\begin{aligned} \beta &= Ah_{1,r}(i\mu^{1/3}\phi) - Bh_{1,i}(i\mu^{1/3}\phi) \\ \Psi &= Bh_{1,r}(i\mu^{1/3}\phi) + Ah_{1,i}(i\mu^{1/3}\phi) \\ \beta' &= -\mu^{1/3}[Bh'_{1,r}(i\mu^{1/3}\phi) + Ah'_{1,i}(i\mu^{1/3}\phi)] \\ \Psi' &= \mu^{1/3}[Ah'_{1,r}(i\mu^{1/3}\phi) - Bh'_{1,i}(i\mu^{1/3}\phi)] \end{aligned} \tag{59}$$

It is noted that with the approximations stated above eq. (42) becomes simply

$$W'' - i\mu\phi W = 0$$

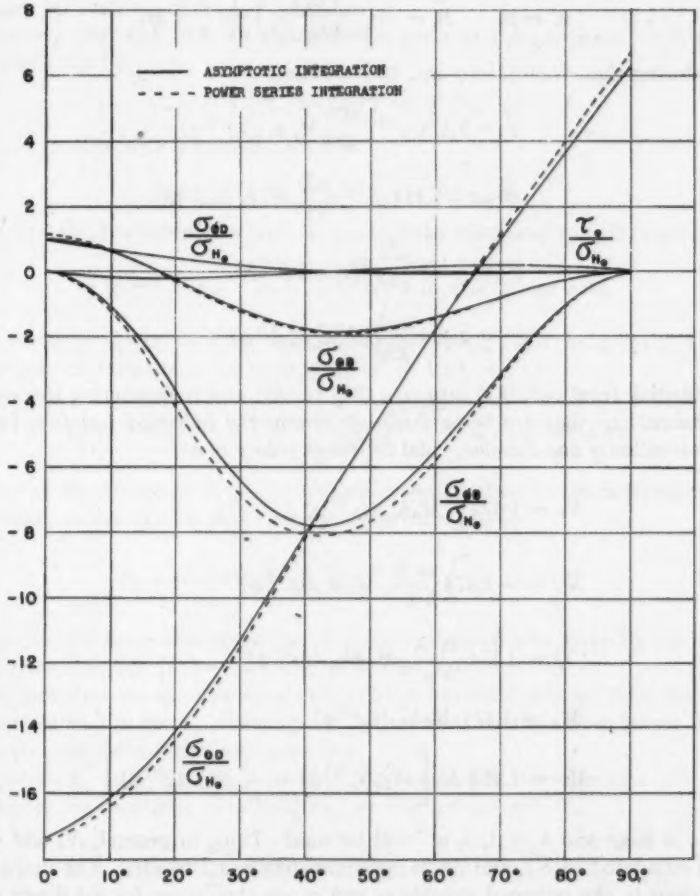

FIG. 6

Equations (59) are merely the result of expressing the solution of this equation explicitly in terms of Bessel functions of order 1/3. That this type of approximation is possible when ϕ is small is indicated by a remark in a paper by E. Meissner [5].

For future reference we list the initial values of the h functions as found in the Harvard Tables [3].

$$h_{1,r}(0) = 0,\ h_{1,i}(0) = -1.0744,\ h'_{1,r}(0) = (.3916)\sqrt{3},\ h'_{1,i}(0) = .3916 \qquad (60)$$

Example 1. If the boundary conditions (57), where now we assume that M_ϕ is proportional to β', are applied to equations (59) one determines, with $h_{1,r}(0) = 0$, that

$$A = 0, \qquad B = -\left(\frac{1.074}{0.392\sqrt{3}}\right)\frac{1}{\mu^{1/3}}\frac{m^2b}{Eh^3}M_0$$

Introducing these values into eqs. (59) we have

$$\begin{aligned}
\beta &= 1.474\,\mu^{-1/3}\frac{m^2b}{Eh^3}M_0\,h_{1,i}(i\mu^{1/3}\phi) \\
\Psi &= -1.474\,\mu^{-1/3}\frac{m^2b}{Eh^3}M_0\,h_{1,r}(i\mu^{1/3}\phi) \\
\beta' &= 1.474\frac{m^2b}{Eh^3}M_0\,h'_{1,r}(i\mu^{1/3}\phi) \\
\Psi' &= 1.474\frac{m^2b}{Eh^3}M_0\,h'_{1,i}(i\mu^{1/3}\phi)
\end{aligned} \tag{61}$$

Substituting from eqs. (61) into eqs. (31) to (35) and remembering the various approximations that are being made we obtain the following formulas for the stress resultants and couples, valid for small values of ϕ:

$$\begin{aligned}
N_\theta &= 1.474\frac{m}{h}M_0\,h'_{1,i}(i\mu^{1/3}\phi) \\
N_\phi &= -1.474\frac{m}{h}\frac{\lambda}{\mu^{1/3}}M_0\,h_{1,r}(i\mu^{1/3}\phi) \\
Q &= 1.474\frac{m}{h}\frac{\lambda}{\mu^{1/3}}M_0\,\phi h_{1,r}(i\mu^{1/3}\phi) \\
M_\phi &= 1.474\,M_0\,h'_{1,r}(i\mu^{1/3}\phi) \\
M_\theta &= 1.474\,M_0\left[\nu h'_{1,r}(i\mu^{1/3}\phi) + \frac{\lambda}{\mu^{1/3}}h_{1,i}(i\mu^{1/3}\phi)\right]
\end{aligned} \tag{62}$$

If μ is large and $\lambda < 1$, $\lambda/\mu^{1/3}$ will be small. Thus, in general, N_ϕ and Q are small compared to N_θ, and M_θ is approximately equal to νM_ϕ. The major contributions to the principal stresses σ_ϕ and σ_θ are then given by the direct stress $\sigma_{\theta D}$ and the bending stress $\sigma_{\phi B}$ (see eqs. (36) and (37)). From the behavior of $h_1(i\mu^{1/3}\phi)$ it follows that the maximum values of these stresses occur at $\phi = 0$. Writing $M_0 = \frac{1}{6}h^2\sigma_{M_0}$ and taking equations (14) and (60) into account, we have

$$\sigma_{\phi B,\max.} = \sigma_{\phi B}(0) = \sigma_{M_0} \tag{63a}$$

$$\sigma_{\theta D,\max.} = \sigma_{\theta D}(0) = \tfrac{1}{3}\sqrt{1-\nu^2}\,\sigma_{M_0} \tag{63b}$$

The maximum value of $\sigma_{\theta D}$ given by eq. (63b) is 11 per cent smaller than that

found by Wissler for $\mu = 6.61$. However, it must be remembered that formulas (63) are obtained by assuming that an edge effect exists. From figure 5 we see that the edge effect is not particularly great for $\mu = 6.61$. As the shell becomes thinner, or as μ becomes larger, the edge effect will be more pronounced and formulas (63) will become more accurate.

From eq. (38) and (62) we find that the horizontal displacement u is given approximately by

$$u = 1.474\ (ma/Eh^2)\ M_0\ h'_{1,i}(i\mu^{1/3}\phi)$$

Just as eq. (63b) was obtained, we have

$$u_{max.} = u(0) = \tfrac{1}{3}\sqrt{1-\nu^2}(a\sigma_{M_0}/E) \tag{64}$$

From eq. (41), neglecting the term in $\epsilon_{\phi M}$ as small compared to β, it follows that

$$w = -1.474\ \mu^{-1/3}(m^2b^2/Eh^3)M_0 \int h_{1,i}(i\mu^{1/3}\phi)\cos\phi\ d\phi.$$

If we let $\delta_V = w(\pi/2) - w(0)$, again set $M_0 = \frac{1}{3}h^2\sigma_{M_0}$, and change to $x = \mu^{1/3}\phi$ as variable of integration, we have, with $m^2 = 12(1-\nu^2)$,

$$\delta_V = -2.95(1-\nu^2)\mu^{-2/3}\frac{b^2\sigma_{M_0}}{Eh}\int_0^{(\pi/2)\mu^{1/3}} h_{1,i}(ix)\cos\phi\ dx$$

In view of the behavior of $h_{1,i}(ix)$ the main contribution to the integral comes from small values of x so that we may approximate δ_V by

$$\delta_V = -0.852\sqrt{1-\nu^2}\,\mu^{1/3}\frac{a\sigma_{M_0}}{E}\int_0^{\infty} h_{1,i}(ix)\ dx \tag{65}$$

In eq. (65) we have also made use of the definition of μ as given by eqs. (14). Comparing (65) and (64) we see that $\delta_V = O[\mu^{1/3}u(0)]$.

The fact that we are able to obtain explicit formulas such as (63), (64) and (65) indicates the great advantage of the present methods over power series solutions even if the latter were practical.

Example 2. Applying the boundary conditions (58) to equations (59) and solving for the constants of integration, we find for example 2,

$$\begin{aligned}
\beta &= -0.930\frac{ma}{Eh^2}H_0[h_{1,r}(i\mu^{1/3}\phi) + \frac{1}{\sqrt{3}}h_{1,i}(i\mu^{1/3}\phi)] \\
\Psi &= \ \ 0.930\frac{ma}{Eh^2}H_0\left[\frac{1}{\sqrt{3}}h_{1,r}(i\mu^{1/3}\phi) - h_{1,i}(i\mu^{1/3}\phi)\right] \\
\beta' &= -0.930\mu^{1/3}\frac{ma}{Eh^2}H_0\left[\frac{1}{\sqrt{3}}h'_{1,r}(i\mu^{1/3}\phi) - h'_{1,i}(i\mu^{1/3}\phi)\right] \\
\Psi' &= -0.930\,\mu^{1/3}\frac{ma}{Eh^2}H_0\left[h'_{1,r}(i\mu^{1/3}\phi) + \frac{1}{\sqrt{3}}h'_{1,i}(i\mu^{1/3}\phi)\right]
\end{aligned} \tag{66}$$

The stress resultants and couples may now be determined as in example 1. Again it is found that only N_θ and M_ϕ are significant. For these quantities we now have

$$N_\theta = -.930\frac{a}{b}\mu^{1/3}\left[h'_{1,r}(i\mu^{1/3}\phi) + \frac{1}{\sqrt{3}}h'_{1,i}(i\mu^{1/3}\phi)\right]H_0$$
$$M_\phi = -.930\frac{ha}{mb}\mu^{1/3}\left[\frac{1}{\sqrt{3}}h'_{1,r}(i\mu^{1/3}\phi) - h'_{1,i}(i\mu^{1/3}\phi)\right]H_0 \tag{67}$$

From the behavior of h'_1 it follows that N_θ is again largest at $\phi = 0$. If we set $H_0 = h\sigma_{H_0}$ we have, using the definition of μ,

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(0) = -1.27(1-\nu^2)^{1/6}(a^2/bh)^{1/3}\sigma_{H_0} \tag{68}$$

The stress couple M_ϕ is zero at $\phi = 0$ and reaches a maximum where $\beta'' = 0$, since we have assumed that M_ϕ is proportional to β'. From eq. (16) with right hand side zero we find that approximately

$$\beta'' = -\mu \sin \phi\Psi.$$

Consequently the maximum of M_ϕ occurs where $\Psi = 0$ or, from eq. (66) with $x = \mu^{1/3}\phi$, where

$$h_{1,r}(ix) - \sqrt{3}h_{1,i}(ix) = 0 \tag{69}$$

The smallest root of this equation is $x = 1.58$. If we designate the value of ϕ where M_ϕ attains a maximum by ϕ_m we have

$$\phi_m = \frac{1.58}{\mu^{1/3}} \tag{70}$$

With $h'_{1,r}(1.58i) = .373$ and $h'_{1,i}(1.58i) = -.123$ it follows from eqs. (67) and (37) that

$$\sigma_{\phi B,\max} = \sigma_{\phi B}(\phi_m) = -.825(1-\nu^2)^{-1/3}(a^2/bh)^{1/3}\sigma_{H_0} \tag{71}$$

The values given by formulas (68), (70) and (71) check only qualitatively with the values obtained more accurately for $\mu = 6.6$ since this value of μ is too small for a true edge effect. It is again emphasized, however, that the explicit formulas obtained become more and more accurate as μ becomes larger. From eq. (70) it is evident that the maximum value of $\sigma_{\phi B}$ occurs for smaller and smaller values of ϕ as μ becomes larger so that for sufficiently large values of μ the original approximation of the sine and cosine functions is valid, as far as determining maximum stresses is concerned.

For the present example we find

$$u_{\max.} = u(0) = -1.27(1-\nu^2)^{1/6}\left(\frac{a^2}{bh}\right)^{1/3}\frac{a\sigma_{H_0}}{E} \tag{72}$$

$$\delta_T \approx 2.13(1-\nu^2)^{1/3}\left(\frac{ab}{h^2}\right)^{1/3}\frac{a\sigma_{H_0}}{E}\int_0^\infty\left[h_{1,r}(ix) + \frac{1}{\sqrt{3}}h_{1,i}(ix)\right]dx \tag{73}$$

so that again $\delta_T = O[\mu^{1/3}u(0)]$.

5. Asymptotic integration of the non-homogeneous equation. If a particular problem involves edge loads with vertical (axial) components or involves the components of load intensity, p_H and p_V, the right side of eq. (27) is not identically zero. It is then necessary to obtain an appropriate particular integral, to be added to the approximate complementary function found in part 3, in order to have a complete solution of the basic differential equation. Ordinarily the solution of the so-called *membrane* theory of shells is taken as an approximation for such particular integrals. For a shell which has a horizontal tangent plane the membrane solutions are not applicable in general and an essentially different approach is required. In this section we attempt to find approximate particular integrals, valid for large μ, in a way which is independent of the exact nature of the loading terms. The method given here was developed jointly with E. Reissner and is also applied in a joint paper on the bending of curved tubes [1].

In eq. (27), with the right hand side given by eqs. (19) and (20), let us now neglect terms in λ^2 and $\nu\lambda$ as small compared to similar terms in μ. Eq. (27) then becomes

$$W'' - \frac{i\mu \sin \phi}{1 + \lambda \sin \phi} W = \mu f(\phi)(1 + \lambda \sin \phi)^{-(1/2)} \tag{74}$$

where

$$f(\phi) = \Omega \cos \phi - (i/\mu)[(1 + \lambda \sin \phi)P_H' + (1 + \nu)\lambda \cos \phi P_H] \tag{75}$$

If the components of load intensity, p_H and p_V, are of the same order of magnitude it follows from eqs. (15) and (30) that Ω and P_H are of the same order also. Consequently the P_H terms in (75), which are multiplied by $1/\mu$, may be neglected unless $\Omega = 0$, in which case they must be retained. An example of the latter situation is a rotating shell subject to centrifugal force.

If we restrict our attention for the moment to regions of a shell away from the points where $\sin \phi$ is zero we find, by assuming that W may be expanded in inverse powers of μ, that equation (74) has a formal particular integral of the form

$$W = if(\phi) \csc \phi(1 + \lambda \sin \phi)^{1/2} + O(1/\mu) \tag{76}$$

This solution ceases to be significant in the neighborhood of $\phi = 0, \pm\pi$ unless the limit of $f(\phi) \csc \phi$ is finite when $\phi \to 0, \pm\pi$. If only the term in Ω is retained in eq. (75) it follows, from eq. (28), that (76) implies

$$\beta = 0 + O(1/\mu) \tag{77a}$$

$$\Psi = \Omega \cot \phi + O(1/\mu) \tag{77b}$$

Equations (77) constitute what is called the *membrane* solution of shell problems. Recalling the definitions of Ψ and Ω, eqs. (15), it is apparent that the membrane solution for toroidal shells is not valid in the neighborhood of $\phi = 0, \pm\pi$ unless the vertical stress resultant V is zero at $\phi = 0, \pm\pi$. If the membrane solution

given by eqs. (77) is valid everywhere then this solution furnishes an approximate particular integral of eq. (74).

Let us now restrict ourselves for the time being to some interval such that $|\phi| \leq \phi_1 < \pi$, and let us assume that $f(0) \neq 0$. The membrane solution, or the formal particular integral given by (76), is then valid only for $\phi = O(1)$. However, when ϕ is small $\sin\phi/(1 + \lambda \sin\phi) \approx \phi$ and $(1 + \lambda \sin\phi)^{-(1/2)} \approx 1$ so that equation (74) may be approximated by

$$W'' - i\mu\phi W = \mu f(\phi) \tag{78}$$

The problem is to find a particular integral which satisfies eq. (78) for small values of ϕ and behaves like (76) when ϕ is not small. We do this by now defining

$$x = \mu^{1/3} \sin\phi \tag{79}$$

and considering the expression

$$W_p(\phi) = \mu^{1/3} f(\phi)(1 + \lambda \sin\phi)^{1/2} T(x) \tag{80}$$

where the function $T(x)$ is yet to be determined. We note that, for large μ, x is large in the region where (76) is valid. If $T(x) \sim i/x$ for large x, then (80) does have the behavior of (76) for $\phi = O(1)$. For small values of ϕ, $W_p \approx \mu^{1/3} f(\phi) T(\mu^{1/3}\phi)$. If we assume that $f(\phi)$ is approximately constant for small values of ϕ, then by substituting W_p into eq. (78) we determine that, for small values of x, the function $T(x)$ must satisfy the equation

$$\frac{d^2T}{dx^2} - ixT = 1 \tag{81}$$

in order for W_p to be a solution of (78). *Thus, $W_p(\phi)$ as given by* (80), *furnishes an approximate particular integral of eq.* (74) *which is valid everywhere, due to the periodicity of $x(\phi)$, provided μ is large and $T(x)$ is a solution of* (81) *asymptotic to i/x for large x.* A complete solution of eq. (74) is represented approximately by

$$W(\phi) = (A + Bi)W_1(\phi) + (C + Di)W_2(\phi) + W_p(\phi) \tag{82}$$

where $W_1(\phi)$ and $W_2(\phi)$ are given by eqs. (51), (55) or (56).

A solution of eq. (81) having the correct asymptotic behavior may be expressed explicitly as

$$T(x) = -\tfrac{3}{2}(ix)^{1/2} S_{0,1/3}[\tfrac{2}{3}(ix)^{3/2}] \tag{83}$$

where $S_{0,1/3}$ is a Lommel function. We now take eq. (83) as the definition of the function $T(x)$. This definition implies the following representation for the real and imaginary parts of $T(x)$ and their derivatives (see [9], p. 345–351).

$$\begin{aligned}
T_r(x) &= s_r(x) + \delta[h_{1,r}(ix) + \sqrt{3}h_{1,i}(ix) + h_{2,r}(ix) - \sqrt{3}h_{2,i}(ix)] \\
T_i(x) &= s_i(x) + \delta[-\sqrt{3}h_{1,r}(ix) + h_{1,i}(ix) + \sqrt{3}h_{2,r}(ix) + h_{2,i}(ix)] \\
T'_r(x) &= s'_r(x) + \delta[\sqrt{3}h'_{1,r}(ix) - h'_{1,i}(ix) - \sqrt{3}h'_{2,r}(ix) - h'_{2,i}(ix)] \\
T'_i(x) &= s'_i(x) + \delta[h'_{1,r}(ix) + \sqrt{3}h'_{1,i}(ix) + h'_{2,i}(ix) - \sqrt{3}h'_{2,i}(ix)]
\end{aligned} \tag{84}$$

where

$$\delta = (\pi/6\sqrt{3})(\tfrac{2}{3})^{1/3} = .346$$

$$s_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) x^{6n+2}}{\Gamma(\frac{1}{3}+2n+1)\Gamma(\frac{2}{3}+2n+1)3^{4n+2}}$$

$$s_i(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) x^{6n+5}}{\Gamma(\frac{1}{3}+2n+2)\Gamma(\frac{2}{3}+2n+2)3^{4n+4}} \tag{85}$$

For large values of x the following asymptotic expansions hold:

$$T_r(x) = \frac{1\cdot 2}{x^4} - \frac{(1\cdot 4\cdot 7)(2\cdot 5\cdot 8)}{x^{10}} + \cdots$$

$$T_i(x) = \frac{1}{x} - \frac{(1\cdot 4)(2\cdot 5)}{x^7} + \cdots \tag{86}$$

The functions $T_r(x)$, $T_i(x)$, $T_r'(x)$ and $T_i'(x)$ are tabulated in table 2 for positive values of x. The values were obtained by numerical integration of eq. (81) subject to the initial conditions implied by equations (83) and (84); namely,

$$T_r(0) = -1.288, \qquad T_i(0) = 0$$
$$T_r'(0) = 0, \qquad T_i'(0) = .939 \tag{87}$$

The values of the functions are obtained for negative values of x by means of the relationships

$$T_r(-x) = T_r(x), \qquad T_i(-x) = -T_i(x)$$
$$T_r'(-x) = -T_r'(x), \qquad T_i'(-x) = T_i'(x) \tag{88}$$

6. Examples involving the solution of the non-homogeneous equation. *Example 3.* We consider first the problem of a particular type of piping expansion bellows known as an "Omega" joint [2]. This joint consists of a toroidal shell welded to two straight sections of pipe as shown in figure 7a. In practice such joints are subjected to anti-symmetrical bending and to internal pressure as well as direct axial expansion or compression. Here we shall discuss only the latter problem of symmetrical axial loads. However, the problem of internal pressure may also be treated by the present methods.

From the symmetry of the problem with respect to the plane $z = 0$ it is apparent that we need consider only the lower half of the shell as in figure 7b. Again from this symmetry, the boundary conditions at $\phi = \pi/2$ will be $\beta = 0$ and $\Psi = 0$. For simplicity we shall assume that these conditions also hold at $\phi = -\pi/2$, although in general the conditions are those of an elastic support at this inner edge. The solution obtained under this assumption is still significant since it may be shown that whatever the actual conditions at $\phi = -(\pi/2)$ only a slight edge effect will occur which will not affect the maximum stresses involved provided μ is sufficiently large.

TABLE II

Values of the functions T_r and T_i and of their derivatives

x	$T_r(x)$	$T_r'(x)$	$T_i(x)$	$T_i'(x)$	x	$T_r(x)$	$T_r'(x)$	$T_i(x)$	$T_i'(x)$
0.00	−1.288	0.000	0.000	0.939	2.55	0.017	0.211	0.579	−0.361
0.05	−1.287	0.050	0.047	0.937	2.60	0.027	0.188	0.561	−0.358
0.10	−1.283	0.100	0.094	0.932	2.65	0.036	0.165	0.543	−0.354
0.15	−1.277	0.149	0.140	0.924	2.70	0.044	0.144	0.525	−0.349
0.20	−1.268	0.197	0.186	0.913	2.75	0.050	0.123	0.508	−0.342
0.25	−1.257	0.245	0.231	0.899	2.80	0.056	0.104	0.491	−0.335
0.30	−1.244	0.292	0.276	0.882	2.85	0.061	0.086	0.475	−0.327
0.35	−1.228	0.337	0.320	0.862	2.90	0.065	0.069	0.459	−0.318
0.40	−1.210	0.380	0.362	0.839	2.95	0.068	0.053	0.443	−0.308
0.45	−1.190	0.422	0.403	0.813	3.00	0.070	0.038	0.428	−0.298
0.50	−1.168	0.462	0.443	0.785	3.05	0.071	0.024	0.413	−0.287
0.55	−1.144	0.500	0.482	0.755	3.10	0.072	0.011	0.399	−0.276
0.60	−1.118	0.535	0.519	0.723	3.15	0.073	0.000	0.386	−0.264
0.65	−1.090	0.568	0.554	0.688	3.20	0.073	−0.010	0.373	−0.253
0.70	−1.061	0.599	0.588	0.652	3.25	0.072	−0.019	0.360	−0.241
0.75	−1.030	0.627	0.619	0.614	3.30	0.071	−0.027	0.348	−0.230
0.80	−0.998	0.652	0.649	0.574	3.35	0.069	−0.034	0.337	−0.218
0.85	−0.965	0.675	0.677	0.534	3.40	0.067	−0.040	0.327	−0.206
0.90	−0.931	0.695	0.702	0.493	3.45	0.065	−0.045	0.317	−0.195
0.95	−0.896	0.712	0.726	0.450	3.50	0.063	−0.050	0.307	−0.184
1.00	−0.860	0.726	0.747	0.407	3.55	0.060	−0.053	0.298	−0.173
1.05	−0.823	0.737	0.767	0.364	3.60	0.058	−0.056	0.290	−0.163
1.10	−0.786	0.745	0.784	0.321	3.65	0.056	−0.058	0.282	−0.152
1.15	−0.749	0.750	0.799	0.278	3.70	0.053	−0.059	0.275	−0.142
1.20	−0.711	0.753	0.812	0.235	3.75	0.050	−0.059	0.268	−0.132
1.25	−0.673	0.753	0.822	0.193	3.80	0.047	−0.059	0.261	−0.123
1.30	−0.636	0.750	0.830	0.151	3.85	0.044	−0.059	0.255	−0.114
1.35	−0.599	0.745	0.836	0.110	3.90	0.041	−0.058	0.250	−0.106
1.40	−0.562	0.737	0.841	0.070	3.95	0.038	−0.057	0.246	−0.098
1.45	−0.525	0.727	0.844	0.031	4.00	0.035	−0.056	0.242	−0.090
1.50	−0.489	0.715	0.846	−0.007	4.05	0.032	−0.054	0.238	−0.083
1.55	−0.454	0.700	0.845	−0.043	4.10	0.029	−0.052	0.234	−0.077
1.60	−0.419	0.683	0.842	−0.078	4.15	0.027	−0.050	0.230	−0.071
1.65	−0.385	0.664	0.837	−0.111	4.20	0.024	−0.048	0.226	−0.066
1.70	−0.352	0.644	0.831	−0.142	4.25	0.022	−0.046	0.223	−0.061
1.75	−0.320	0.623	0.823	−0.171	4.30	0.020	−0.043	0.220	−0.057
1.80	−0.290	0.601	0.814	−0.198	4.35	0.018	−0.040	0.218	−0.053
1.85	−0.261	0.578	0.804	−0.222	4.40	0.016	−0.038	0.215	−0.049
1.90	−0.232	0.554	0.793	−0.244	4.45	0.014	−0.035	0.213	−0.046
1.95	−0.204	0.529	0.780	−0.264	4.50	0.012	−0.032	0.210	−0.043
2.00	−0.178	0.503	0.766	−0.282	4.55	0.011	−0.030	0.208	−0.040
2.05	−0.153	0.476	0.751	−0.298	4.60	0.009	−0.027	0.206	−0.038
2.10	−0.130	0.449	0.736	−0.313	4.65	0.008	−0.025	0.205	−0.036
2.15	−0.108	0.421	0.720	−0.326	4.70	0.007	−0.022	0.203	−0.034
2.20	−0.088	0.394	0.703	−0.337	4.75	0.006	−0.020	0.201	−0.033
2.25	−0.069	0.366	0.686	−0.346	4.80	0.005	−0.018	0.200	−0.031
2.30	−0.051	0.339	0.669	−0.353	4.85	0.004	−0.016	0.198	−0.030
2.35	−0.035	0.312	0.651	−0.358	4.90	0.003	−0.014	0.197	−0.029
2.40	−0.020	0.286	0.633	−0.361	4.95	0.003	−0.012	0.195	−0.029
2.45	−0.006	0.260	0.615	−0.362	5.00	0.002	−0.010	0.194	−0.028
2.50	0.006	0.235	0.597	−0.362					

The components of load intensity, p_N and p_T, are zero. Hence the quantity Ω is now a constant and may be taken as

$$\Omega_0 = \frac{ma}{Eh^2} V_0 = \frac{ma}{Eh} \sigma_{T_0}$$

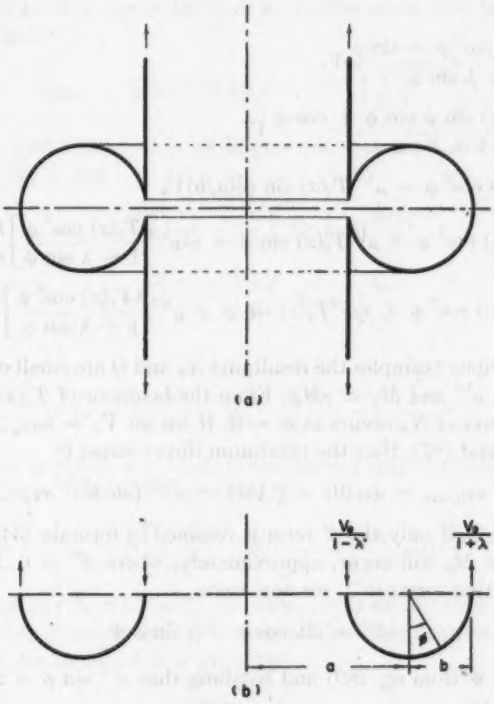

FIG. 7

According to eqs. (75) and (80) a particular solution of eq. (74) is now approximated by

$$W_p = \mu^{1/3}\Omega_0 T(x) \cos\phi(1 + \lambda \sin\phi)^{1/2}$$

Using eqs. (29) the corresponding formulas for β, Ψ and their derivatives become

$$\beta = \mu^{1/3} T_r(x) \cos\phi\, \Omega_0 \tag{89a}$$

$$\Psi = \mu^{1/3} T_i(x) \cos\phi\, \Omega_0 \tag{89b}$$

$$\beta' = [\mu^{2/3} T_r'(x) \cos^2\phi - \mu^{1/3} T_r(x) \sin\phi]\Omega_0 \tag{89c}$$

$$\Psi' = [\mu^{2/3} T_i'(x) \cos^2\phi - \mu^{1/3} T_i(x) \sin\phi]\Omega_0 \tag{89d}$$

It is now observed that formulas (89) satisfy the assumed boundary conditions at $\phi = \pi/2$, $-(\pi/2)$ so that for this particular problem no solution of the homogeneous equation need be superimposed.

Substituting from eqs. (89) into eqs. (31) to (35) we obtain the following formulas for the stress resultants and couples, where Ω_0 is now replaced by $(ma/Eh^2)V_0$.

$$N_\phi = \frac{\mu^{1/3}T_i(x)\cos^2\phi + \sin\phi}{1 + \lambda\sin\phi}V_0 \tag{90a}$$

$$Q = \frac{-\mu^{1/3}T_i(x)\sin\phi\cos\phi + \cos\phi}{1 + \lambda\sin\phi}V_0 \tag{90b}$$

$$N_\theta = [\mu^{2/3}T_i'(x)\cos^2\phi - \mu^{1/3}T_i(x)\sin\phi](a/b)V_0 \tag{90c}$$

$$M_\phi = \left[\mu^{2/3}T_r'(x)\cos^2\phi - \mu^{1/3}T_r(x)\sin\phi + \nu\lambda\mu^{1/3}\frac{T_r(x)\cos^2\phi}{1 + \lambda\sin\phi}\right]\frac{ha}{mb}V_0 \tag{90d}$$

$$M_\theta = \left[\nu\mu^{2/3}T_r'(x)\cos^2\phi - \nu\mu^{1/3}T_r(x)\sin\phi + \mu^{1/3}\frac{\lambda T_r(x)\cos^2\phi}{1 + \lambda\sin\phi}\right]\frac{ha}{mb}V_0 \tag{90e}$$

As in the previous examples the resultants N_ϕ and Q are small compared to N_θ by a factor of $\lambda/\mu^{1/3}$ and $M_\theta \approx \nu M_\phi$. From the behavior of $T_i'(x)$ it is apparent that the maximum of N_θ occurs at $\phi = 0$. If we set $V_0 = h\sigma_{V_0}$, we find, using eqs. (14), (36), and (87), that the maximum direct stress is

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(0) = 2.15(1 - \nu^2)^{1/3}(ab/h^2)^{1/3}\sigma_{V_0} \tag{91}$$

As in example 2, if only the β' term is retained in formula (34) for M_ϕ, then the maximum of M_ϕ will occur, approximately, where $\beta'' = 0$. From eqs. (16) and (19), neglecting terms in λ, we now have

$$\beta'' = \mu\Omega_0\cos\phi - \mu\sin\phi\,\Psi$$

Substituting for Ψ from eq. (89) and recalling that $\mu^{1/3}\sin\phi = x$, the equation $\beta'' = 0$ reduces to

$$xT_i(x) = 1$$

The smallest numerical values of x satisfying this equation are $x = \pm 1.225$. Denoting the angle where M_ϕ attains its maximum by ϕ_m, it follows that

$$\phi_m = \pm\sin^{-1}\frac{1.225}{\mu^{1/3}} \approx \pm\frac{1.225}{\mu^{1/3}} \tag{92}$$

Retaining only the first term in eq. (90d) for M_ϕ we obtain for the maximum value of the bending stress $\sigma_{\phi B}$, defined by eqs. (36), the formula

$$\sigma_{\phi B,\max} = \sigma_{\phi B}(\phi_m) = \pm 2.99(1 - \nu^2)^{-1/6}(ab/h^2)^{1/3}\sigma_{V_0} \tag{93}$$

where the sign corresponds to the sign of ϕ_m as given by (92). For this problem, as well as for example 2 of the previous section, the maximum bending stress occurs closer and closer to $\phi = 0$ as μ becomes larger.

The horizontal displacement u follows from eqs. (90) and (38), where we now neglect N_ϕ in comparison with N_θ.

$$u = (1 + \lambda \sin \phi)[\mu^{2/3} T_i'(x) \cos^2 \phi - \mu^{1/3} T_i(x) \sin \phi] \frac{a^2}{bh} \frac{V_0}{E}$$

The largest value of u occurs for $\phi = 0$. In the same way that eq. (91) was obtained, we find

$$u_{\max.} = u(0) = 2.15(1 - \nu^2)^{1/3} \left(\frac{a^2}{bh}\right)^{2/3} \frac{b\sigma_{V_0}}{E} \tag{94}$$

Using the fact that $T_i(x) \sim 1/x$ for large x where $x = \mu^{1/3} \sin \phi$, it follows from the expression for u that

$$u\left(\frac{\pi}{2}\right) = -\left(\frac{a}{b}\right)^2 (1 + \lambda) \frac{b\sigma_{V_0}}{E}$$

$$u\left(-\frac{\pi}{2}\right) = -\left(\frac{a}{b}\right)^2 (1 - \lambda) \frac{b\sigma_{V_0}}{E}$$

Hence

$$\delta_H = u\left(\frac{\pi}{2}\right) - u\left(-\frac{\pi}{2}\right) = -2 \frac{a}{b} \frac{b\sigma_{V_0}}{E} \tag{95}$$

This means that the shell as a whole is compressed only slightly in the horizontal direction. However, from (94) and eqs. (14),

$$\delta_H = O\left[\left(\frac{h^2}{ab}\right)^{1/3} u(0)\right] = O\left[\frac{\lambda}{\mu^{2/3}} u(0)\right]$$

so that the quantity δ_H may be of the same order as the error in the entire approximation. Consequently eq. (95) may not be quantitatively valid.

The vertical displacement w is given by

$$w = -\mu^{1/3} \frac{mab}{Eh} \sigma_{V_0} \int T_r(x) \cos^2 \phi \, d\phi$$

according to eq. (41), where $\epsilon_{\phi M}$ is neglected in comparison with β and β is taken from eq. (89) with $\Omega_0 = (ma/Eh)\sigma_{V_0}$. As was done in example 1, eq. (65), we write

$$\delta_V = w\left(\frac{\pi}{2}\right) - w(0) \approx -\frac{mab}{Eh} \sigma_{V_0} \int_0^\infty T_r(x) \, dx$$

The integral $\int_0^\infty T_r(x) \, dx$ has been calculated numerically and to three place accuracy the value is $-(\pi/2)$. Consequently

$$\delta_V = 5.44(1 - \nu^2)^{1/3} \frac{a}{h} \frac{b\sigma_{V_0}}{E} \tag{96}$$

The great flexibility of a toroidal joint such as we are considering is indicated by formula (96), since the quantity $b\sigma_{T_0}/E$ may be interpreted as the extension in a straight pipe section of length b. From the symmetry of the present problem it follows that the distance the two inner edges in fig. 9 will spread apart (or compress) is equal to $4\,\delta_T$.

It is interesting to note that mathematically the particular problem just solved is completely equivalent to the problem of the bending of a curved tube considered from the point of view of shell theory (see [1]).

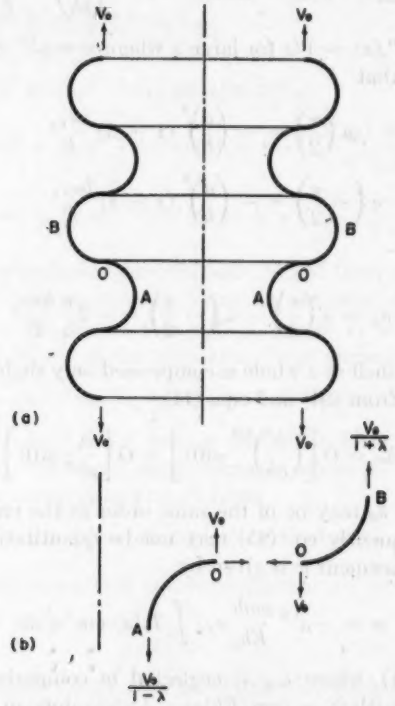

FIG. 8

Example 4: Corrugated pipe subject to axial load. Consider a pipe with a cross section as shown in figure 8a subject to a uniform axial load. Away from the ends of the pipe the state of stress may be assumed to be periodic in the axial direction so that it is necessary to obtain a solution only for a portion of the shell such as AOB. The section AO is taken as the portion of a toroidal shell corresponding to the interval $\pi \leq \phi \leq 3\pi/2$; the section OB corresponds to the interval $0 \leq \phi \leq \pi/2$. These two sections are shown broken apart in figure 8b. For the interval $0 \leq \phi \leq \pi/2$ the particular solution given by formula (80) again implies eqs. (89) for β, Ψ and their derivatives. For the interval $\pi \leq \phi \leq 3\pi/2$ the same equations are valid except that V_0 must be replaced by $-V_0$

since the parameter ϕ now increases in the opposite direction and we have assumed that the stress resultant V is positive upwards at the edge where ϕ is largest. The formulas thus defined for the entire section AOB give a continuous representation for β, Ψ and β' but imply that Ψ' is discontinuous at the point O. Consequently it is necessary to superimpose a solution of the homogeneous differential equation in order to obtain a continuous solution of the entire problem.

The significance of the superposition of a homogeneous solution may be interpreted as follows. Equation (89b) gives $\Psi = 0$ at point O implying that no horizontal force is acting. However if the sections AO and OB are separated then the forces V_0 will tend to rotate the edges at point O so as to cause horizontal displacements in opposite directions. Thus it is necessary to superimpose a horizontal force at point O in order to pull the sections together again. One might also suppose it necessary to superimpose a moment but since the rotations of the edges at point O produced either by the vertical or the horizontal forces are of the same character and magnitude the moment M_ϕ is zero at this point, at least within the error of the approximation being made here. It may be verified that this condition is satisfied by equations (89) if we assume M_ϕ proportional to β'.

Let us now consider only the section OB since to within terms in λ the solution will be symmetrical with respect to the point O. The boundary conditions at $\phi = \pi/2$ become $\beta = 0$ and $\Psi = 0$ due to symmetry conditions for the entire shell. Since it has been shown in example 2 that the horizontal force which must be applied at $\phi = 0$ has no significant effect on the conditions at $\phi = \pi/2$ when μ is sufficiently large, the problem may be solved by superimposing the solutions of examples 2 and 3 in such a way as to make the horizontal displacement $u = 0$ at $\phi = 0$.

Assuming that u is proportional to Ψ' we have

$$\Psi'(0) = \Psi'_p(0) + \Psi_2(0) = 0 \tag{97}$$

where Ψ'_p is given by eq. (89d) and Ψ_2 by eq. (66d). Introducing the values given by these equations, and eqs. (60) and (87), we have

$$.939\mu^{3/2}\frac{ma}{Eh^2}V_0 - .930\left(.678 + \frac{.392}{\sqrt{3}}\right)\mu^{1/2}\frac{ma}{Eh^2}H_0 = 0$$

or

$$H_0 = 1.115\,\mu^{1/2}V_0 \tag{98}$$

Using this value of H_0 equations for β, Ψ and their derivatives are obtained by combining eqs. (66) and (89). The stress resultants and couples and the displacements follow from eqs. (31) to (35), (38) and (41) as in previous examples. Again, as far as maximum stresses are concerned, only N_θ and M_ϕ are significant. These quantities are given by

$$N_\theta = \frac{a}{b}V_0\left\{\mu^{2/3}T'_4(x)\cos^2\phi - \mu^{1/3}T_4(x)\sin\phi - 1.04\mu^{2/3}\left[h'_{1,r}(i\mu^{1/3}\phi) + \frac{1}{\sqrt{3}}h'_{1,i}(i\mu^{1/3}\phi)\right]\right\} \tag{99}$$

$$M_\phi = \frac{ha}{mb} V_0 \left\{ \mu^{2/3} T_r'(x) \cos^2 \phi - \mu^{1/3} T_r(x) \sin \phi - 1.04\mu^{2/3} \left[\frac{1}{\sqrt{3}} h_{1,r}'(i\mu^{1/3}\phi) - h_{1,i}'(i\mu^{1/3}\phi) \right] \right\} \tag{100}$$

Let ϕ_{m_1} and ϕ_{m_2} denote the values of ϕ where N_θ and M_ϕ have maximums. Since N_θ is proportional to Ψ' and since, from eq. (17),

$$\Psi'' \approx \mu \sin \phi \, \beta$$

it follows that ϕ_{m_1} is a solution of the equation $\beta = 0$. This equation implies, with $\cos \phi \approx 1$ and $\sin \phi \approx \phi$,

$$T_r(x) - 1.04[h_{1,r}(ix) + (1/\sqrt{3}) h_{1,i}(ix)] = 0 \tag{101}$$

The smallest root of (101) is $x = 1.89$, hence

$$\phi_{m_1} = 1.89/\mu^{1/3} \tag{102}$$

Using eqs. (14), (36) and (99) we then have, with $V_0 = h\sigma_{r_0}$,

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(\phi_{m_1}) = -.925(1 - \nu^2)^{1/3}(ab/h^2)^{1/3}\sigma_{r_0} \tag{103}$$

As in example 2 the location of the maximum of M_ϕ is determined by the equation

$$\beta'' = \mu(\Omega \cos \phi - \Psi \sin \phi) = 0$$

Using equations (66b) and (87b) this equation in turn implies that

$$1 - xT_i(x) + 1.04x[(1/\sqrt{3}) h_{1,r}(ix) - h_{1,i}(ix)] = 0 \tag{104}$$

The smallest root of eq. (104) is $x = 1.35$, hence

$$\phi_{m_2} = 1.35/\mu^{1/3} \tag{105}$$

It then follows that

$$\sigma_{\phi B,\max} = \sigma_{\phi B}(\phi_{m_2}) = 1.63(1 - \nu^2)^{-1/6}(ab/h^2)^{1/3}\sigma_{r_0} \tag{106}$$

We note that this value for the maximum of the bending stress $\sigma_{\phi B}$ is nearly twice as large as the maximum value of the direct stress $\sigma_{\theta D}$ given by (103).

Equation (102) shows that for sufficiently large values of μ the maximum $\sigma_{\theta D}$ stress does not occur in the outermost fibers ($\phi = \pi/2$) as might be expected, but is located nearer and nearer to $\phi = 0$ as μ becomes larger. The same situation is true for the horizontal displacement u since, according to eqs. (36 and 38), u is approximately proportional to $\sigma_{\theta D}$. In particular, we find

$$u_{\max.} \approx u(\phi_{m_1}) = -.925(1 - \nu^2)^{1/3} \left(\frac{a^2}{bh} \right)^{2/3} \frac{b\sigma_{r_0}}{E} \tag{107}$$

The vertical displacement w is again found by using eq. (41). Neglecting the term in $\epsilon_{\phi M}$, we have, with β from eqs. (66a) and (89a) and H_0 from (98),

$$w = -\mu^{1/3} \frac{mab}{Eh} \sigma_{r_0} \int \left\{ T_r(x) \cos \phi - 1.04 \left[h_{1,r}(i\mu^{1/3}\phi) + \frac{1}{\sqrt{3}} h_{1,i}(i\mu^{1/3}\phi) \right] \right\} d\phi$$

As in previous examples we write

$$\delta_V = w(\pi/2) - w(0)$$

$$\approx -3.46\sqrt{1-\nu^2}\,\frac{a}{h}\,\frac{b\sigma_{V_0}}{E}\int_0^\infty \left\{T_r(x) - 1.04\left[h_{1,r}(ix) + \frac{1}{\sqrt{3}}h_{1,i}(ix)\right]\right\}dx \tag{108}$$

Eq. (108) shows that δ_V is of the same order of magnitude for this problem as it was for example 3. If n is the number of toroidal rings or semi-circular cross sections which make up the corrugated pipe then for the total vertical deflection Δ_V of the pipe we have

$$\Delta_V = O\left(\frac{a}{h}\cdot 2n\,\frac{b\sigma_{V_0}}{E}\right) \tag{109}$$

where $2n\,b\sigma_{V_0}/E$ represents the axial deflection which would occur in a straight pipe of the same length.

For the section AO it may be shown that the maximum direct stress $\sigma_{\phi D}$ and the maximum bending stress $\sigma_{\phi B}$ occur at similar locations but are opposite in sign. The sign of the displacements u and w also changes.

Example 5. Corrugated cylinder subject to internal pressure. We now consider the same type of shell as in example 4 subject to a uniform internal pressure distribution. If the pressure is denoted by p we find for a section such as OB that the components of load intensity p_H and p_V are given by

$$p_H = p\sin\phi, \qquad p_V = -p\cos\phi \tag{110}$$

From equations (15) and (30) we find

$$\begin{aligned}\Omega &= \frac{mabp}{Eh^2}\int_0^\phi (1+\lambda\sin\phi)\cos\phi\,d\phi \\ &= \frac{mabp}{Eh^2}\sin\phi\left(1+\frac{\lambda}{2}\sin\phi\right)\end{aligned} \tag{111}$$

If the problem were that of a complete circular torus under pressure the membrane solution, which corresponds to taking (76) as the solution of (74), would be valid. This solution yields (see [8], page 366)

$$N_\theta = \frac{bp}{2}, \qquad N_\phi = bp\,\frac{1+\frac{1}{2}\lambda\sin\phi}{1+\lambda\sin\phi}, \qquad \beta, M_\phi, M_\theta = 0 \tag{112}$$

However, for the present problem there is a bending effect at the point O which introduces stresses much larger than those given by the membrane solution.

Again we consider a section such as AOB broken apart at the point O. We may expect that for a sufficiently thin shell the membrane solution will be valid in the neighborhood of points such as A and B. At the point O the membrane solution (112) implies forces acting at the edges as shown in fig. 9, since for the section OB the membrane solution implies that N_θ and N_ϕ are positive, while for the section AO (considered a portion of a complete toroidal shell) the pressure is of the opposite character which implies that N_θ and N_ϕ are negative. The actual situation must then be that of no horizontal force at point O. This may be

accomplished by superimposing the solution of example 2, as was done in the previous example. However, the two horizontal forces which it is necessary to superimpose on the two edges at point O will be of opposite character and, hence, will produce rotations of the opposite character at this point. Consequently it is necessary to also superimpose a moment at these edges, by means of the solution of example 1, which will make the net rotation β zero at point O.

Considering only the section OB, as in example 4, the membrane solution is represented, according to eqs. (77) and (111) by

$$\beta_m = 0 \quad (113a), \qquad \Psi_m = \frac{mabp}{Eh^2} \cos\phi \left(1 + \frac{\lambda}{2} \sin\phi\right) \tag{113b}$$

$$\beta_m' = 0 \quad (113c), \qquad \Psi_m' = -\frac{mabp}{Eh^2} \left(\sin\phi - \frac{\lambda}{2} \cos 2\phi\right) \tag{113d}$$

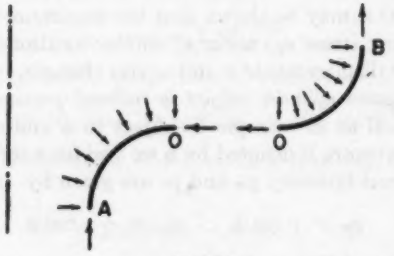

FIG. 9

Denoting the solutions of example 1 and 2 by subscripts 1 and 2, the conditions that $\beta = 0$ and $\Psi = 0$ (no horizontal force) at $\phi = 0$ become

$$\beta(0) = \beta_m(0) + \beta_1(0) + \beta_2(0) = 0 \tag{114a}$$

$$\Psi(0) = \Psi_m(0) + \Psi_1(0) + \Psi_2(0) = 0 \tag{114b}$$

Substituting from eqs. (61), (66) and (113) we have

$$0 - \frac{1.074}{.392\sqrt{3}} \mu^{-1/3} \frac{m^2 b}{Eh^3} M_0 + \frac{1}{\sqrt{3}} \frac{ma}{Eh^2} H_0 = 0 \tag{115a}$$

$$\frac{mabp}{Eh^2} + \qquad 0 \qquad + \qquad \frac{ma}{Eh^2} H_0 = 0 \tag{115b}$$

Eq. (115b) implies

$$H_0 = -bp \tag{116}$$

From eq. (115a) we then have

$$M_0 = -\frac{.392}{1.074} \mu^{1/3} \frac{ah}{m} p \tag{117}$$

With these values of H_0 and M_0 we now combine equations (61), (66), and(113) to obtain

$$\beta = .930\,\frac{mabp}{Eh^2}\,h_{1,r}(i\mu^{1/3}\phi) \tag{118a}$$

$$\Psi = \frac{mabp}{Eh^2}\left[\cos\phi\left(1+\frac{\lambda}{2}\sin\phi\right) + .930h_{1,i}(i\mu^{1/3}\phi)\right] \tag{118b}$$

$$\beta' = -.930\mu^{1/3}\,\frac{mabp}{Eh^2}\,h'_{1,i}(i\mu^{1/3}\phi) \tag{118c}$$

$$\Psi' = \frac{mabp}{Eh^2}\left[-\sin\phi + \frac{\lambda}{2}\cos 2\phi + .930\mu^{1/3}h'_{1,r}(i\mu^{1/3}\phi)\right] \tag{118d}$$

All other quantities may now be obtained as in previous examples. We again find that the maximum stresses are due essentiallyto $\sigma_{\theta D}$ and $\sigma_{\phi B}$. In the same manner as in previous examples we find:

$$\sigma_{\theta D,\max} = \sigma_{\theta D}(0) = .955(1-\nu^2)^{1/6}\left(\frac{ab}{h^2}\right)^{2/3} p \tag{119}$$

$$\sigma_{\phi B,\max} = \sigma_{\phi B}(0) = -.955(1-\nu^2)^{-1/3}\left(\frac{ab}{h^2}\right)^{2/3} p \tag{120}$$

$$u_{\max.} = u(0) = .955(1-\nu^2)^{1/6}\left(\frac{a}{h}\right)^{4/3}\left(\frac{a}{b}\right)^{1/3}\frac{bp}{E} \tag{121}$$

$$\delta_V = w\left(\frac{\pi}{2}\right) - w(0) = -2.13(1-\nu^2)^{1/3}\left(\frac{a}{h}\right)^{4/3}\left(\frac{b}{h}\right)^{1/3}\frac{bp}{E}\int_0^\infty h_{1,r}(ix)\,dx \tag{122}$$

If terms in λ are neglected then for the whole section AOB we find that β and Ψ are odd functions about the point O, β' and Ψ' even functions. Hence the displacement w will be in the opposite direction for the section AO with the result that the total vertical deflection or spread Δ_V of the corrugated cylinder is zero within the order of the approximation being made. The membrane stresses and the displacement u at the points A and B are of the order $\lambda/\mu^{1/3}$ times the maximum values given at point O by formulas (119), (120), and (121).

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AN APPLICATION OF AUTO-CORRELATION ANALYSIS*

By Ernest R. Kretzmer

1. Introduction. Pulse-time modulation (PTM) is the generic term for those types of pulse modulation in which information is conveyed purely through the timing of pulse edges. Pulse-duration modulation (PDM) and pulse-position modulation (PPM) are the two most useful particular forms of PTM. A study of the interference characteristics of such systems reveals that interference, such as may be caused by another station, produces certain effects which are readily analyzed by the auto-correlation method. The origin and nature of these effects will be described in the following section, which is intended only to show the reasons for the subsequent analyses rather than to present a complete discussion of interference effects.[1]

2. Random Effects Caused by Interference in PTM Systems. A process of great importance in optimizing the interference characteristics of PTM systems is the so-called slicing process which is carried out directly after detection of the r-f pulses. It is illustrated in Fig. 1-a. At the instant at which a pulse edge at the slicer input passes the slicing level, a new, much steeper pulse edge appears at the slicer output. An interfering signal or noise may advance or delay this instant (Fig. 1-b), thereby shifting the pulse edges in the slicer output from their correct positions. This effect may be called the "time-shift effect"; the sense and magnitude of the time shift depend on the r-f phase difference between the desired and interfering signals. This phase difference usually changes randomly from one pulse to the next (as a result of incoherence between the desired and undesired signals). Consequently, the time shifts also vary randomly from pulse to pulse; they have a definite probability distribution which depends on such variables as type and magnitude of interfering signal, shape of desired-pulse edges before slicing, the relative height of slicing level. The analysis of the time-shift effect will be carried out in general terms, so that any probability distribution can be substituted into the final result.

A more drastic effect is encountered if the interference is so strong as to make possible a substantial degree of cancellation between the desired and interfering signals. Some of the resultants are then too small to reach the slicing level (r-f phase difference near 180°) and will consequently fail to produce a pulse at the slicer output. (If the interference is intermittent, portions of pulses rather than entire pulses may be randomly missing.)

The purpose of the analyses which follow is to evaluate the ultimate manifestations of the effects described above. Both of these cause random noise in the

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[1] A more detailed discussion is given in Research Laboratory of Electronics Technical Report No. 92, M.I.T., "Interference Characteristics of Pulse-Time Modulation," by the author.

receiver output; the power spectrum and audio power of this noise will be determined. The effect of randomly missing pulses may be regarded as a special case of the time-shift effect and requires only brief treatment following the general analysis of the time-shift effect.

3. Terminology and Notation. The time function to be analyzed is a sequence of rectangular pulses, the edges of which are subjected to random time shifts. If only one edge, e.g., the trailing edge of each pulse is subject to this effect, one may speak of one-edge time shifts; this is the simplest possibility and will be considered first. With both edges of each pulse affected, one may speak of two-edge time shifts. There are several types of two-edge time shifts, depending on the relation, if any, between the shifts of the two edges of any pulse.

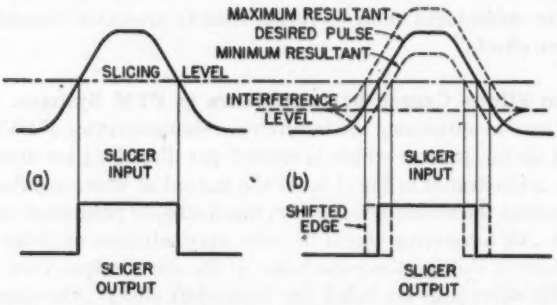

FIG. 1. a—Slicing process in the absence of interference. b—Slicing process in the presence of interference, showing time shifts of pulse edges.

The notation used is as follows:

E = pulse amplitude in volts
T = repetition period of pulses
d = average pulse duration
D = duty factor = d/T
x = time shift of a pulse edge from its normal time position
x_0 = maximum value of x
$\overline{\Delta t}$ = standard deviation of time shifts = $\overline{x} = \sqrt{\overline{x^2} - \bar{x}^2}$
$p(u)$ = probability density distribution of time shifts; $p(u)du$ = probability of finding x between u and $u + du$
$\varphi_{11}(\tau)$ = auto-correlation of time function $f_1(t)$ = $\operatorname{Lim}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+\tau)\,dt$
$\varphi_{\Delta}(\tau)$ = aperiodic component of the auto-correlation $\varphi_{11}(\tau)$
$\Phi_{11}(\omega)$ = power-density spectrum of time function $f_1(t)$, equal to the Fourier transform of $\varphi_{11}(\tau)$
$\Phi_{\Delta}(\omega)$ = continuous component of power-density spectrum $\Phi_{11}(\omega)$, equal to the Fourier transform of $\varphi_{\Delta}(\tau)$.

4. Derivation of Auto-Correlation and Power Spectrum. Beginning with no knowledge other than that of the Fourier transformation, one can derive the

general expressions for the auto-correlation and power spectrum, as well as the relation between these, in a few steps. If $g_1(\omega)$ and $g_2(\omega)$ are the Fourier transforms of the analytic functions $f_1(\tau)$ and $f_2(\tau)$, respectively, it can be shown by Fourier transform methods that

$$\int_{-\infty}^{\infty} g_1(\omega)\overline{g_2(\omega)}e^{j\omega\tau}\,d\omega = (1/2\pi)\int_{-\infty}^{\infty} f_1(t)f_2(t+\tau)\,dt. \tag{1}$$

Letting $f_2(\tau) = f_1(\tau)$, $g_2(\omega) = g_1(\omega)$, one obtains

$$\int_{-\infty}^{\infty} |\, g_1(\omega)\,|^2\, e^{j\omega\tau}\,d\omega = (1/2\pi)\int_{-\infty}^{\infty} f_1(t)f_1(t+\tau)\,dt. \tag{2}$$

This relation is not valid if $f_1(\tau)$ is a random function, infinite in extent along the τ-axis, since both integrals of (2) would be infinite. However, they can be made finite by dividing each side by a very large period, resulting in (3).

$$\operatorname*{Lim}_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} |\, g_1(\omega)\,|^2\, e^{j\omega\tau}\,d\omega = \operatorname*{Lim}_{T\to\infty}\frac{1}{2\pi}\frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+\tau)\,dt. \tag{3}$$

In (2), where $f_1(\tau)$ is an analytic function, such as a voltage pulse, $g_1(\omega)$ represents the corresponding spectral distribution in volts per radian/sec. In (3), where $f_1(\tau)$ is a random voltage function, $g_1(\omega)$ does not exist, but the quantity $\operatorname*{Lim}_{T\to\infty} \pi\,|\,g_1(\omega)\,|^2/T$ does exist; it has the dimensions volt^2 per radian/sec or watts per radian/sec if a one-ohm resistance level is assumed. This quantity therefore gives the amount of power in a one-radian-per-second bandwidth as a function of radian frequency and is called the power-density spectrum, or simply power spectrum, denoted by $\Phi_{11}(\omega)$. Substitution of this in (3) yields

$$\int_{-\infty}^{\infty} \Phi_{11}(\omega)e^{j\omega\tau}\,d\omega = \operatorname*{Lim}_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+\tau)\,dt. \tag{4}$$

The left-hand side of (4) is the inverse Fourier transform of the power spectrum $\Phi_{11}(\omega)$; it equals the quantity on the right-hand side, which is known as the auto-correlation of $f_1(t)$, denoted by $\varphi_{11}(\tau)$. Conversely, it follows that the power spectrum equals the transform of the auto-correlation:

$$\Phi_{11}(\omega) = (1/2\pi)\int_{-\infty}^{\infty} \varphi_{11}(\tau)e^{-j\omega\tau}\,d\tau. \tag{5}$$

In order to find the power spectrum of $f_1(t)$, one first determines the auto-correlation

$$\varphi_{11}(\tau) = \operatorname*{Lim}_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+\tau)\,dt \tag{6}$$

and then obtains its Fourier transform. Expressed in words, the auto-correlation is the long-time average of the product of the time function and the same function shifted by an amount τ. Each multiplication and averaging process would furnish only a single point in the auto-correlation curve. It will be noted from either (5) or (6) that the auto-correlation reduces to the total power for $\tau = 0$.

5. Auto-Correlation of Periodic Rectangular Pulses. Since a train of rectangular pulses is a two-valued time function (its instantaneous value being either E or zero), its auto-correlation (in the absence of interference) is easily evaluated. The multiplication of the function by its shifted replica merely amounts to noting the amount of overlap between the shifted and unshifted pulses. Clearly, this overlap varies linearly with the shift τ, and the auto-correlation is therefore a periodic sequence of triangles, as shown in Fig. 2-a. Because of the periodicity, the overlap condition for any one pulse is the same as for all others, so that only a single pulse-repetition period need be examined. The average value of the product of the two pulse trains is $(E^2/T)\,d$ for $\tau = 0$ or $\tau = nT$ (n = any integer), when they coincide precisely. Changing τ produces a linear decrease until zero is reached, for $|\,\tau + nT\,| = d$. The auto-correlation is therefore given by (7).

$$\varphi_{11}(\tau) = (E^2/T)\,(d - |\,\tau + nT\,|\,); \qquad |\,\tau + nT\,| \leqq d. \qquad (7)$$

6. Auto-Correlation of Rectangular Pulses with One-Edge Time Shifts. Repeating the reasoning used to obtain the auto-correlation for unmodulated pulses, one

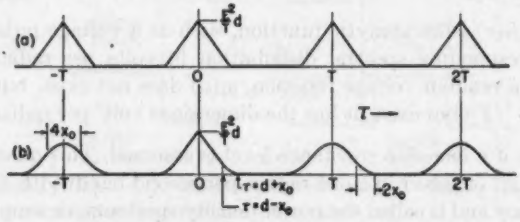

FIG. 2. a—Auto-correlation of periodic rectangular pulses. b—Auto-correlation of rectangular pulses with one-edge time shifts.

can readily find the qualitative effect of a small random time modulation on that auto-correlation. Since the time-modulated edges are blurred in the sense that they deviate randomly from periodicity, the sharp corners of the auto-correlation triangles are rounded off as shown in Fig. 2-b. The rounding at the bottom of each triangle follows from the fact that, as τ is varied, overlap does not cease simultaneously for all pairs of pulses, since no two pulses are of equal duration. Similarly, the curvature at the top of each triangle is produced because some short pulses overlapped by longer pulses contribute constant amounts of overlap over finite ranges of τ; this, however, cannot happen in the case of the correlation triangle centered at $\tau = 0$, since, for small values of τ, each pulse overlaps only its own shifted replica which must obviously be of identical duration. The result is that the "corner" at $\tau = 0$ is the only one not affected by the random modulation. Consequently, the auto-correlation is no longer purely periodic; however, it can be resolved into two components, as shown in Fig. 3. The component shown at the top is periodic, and its transform is the spectrum of the periodic part of the pulse train; the component shown at the bottom of Fig. 3 is aperiodic, and its transform is the spectrum of the random part of the pulse

train. The aperiodic component is the difference between that lobe of the auto-correlation centered about $\tau = 0$ and any other lobe. It is denoted by $\varphi_\Delta(\tau)$ and is dependent only on the shape of the rounded peaks of the auto-correlation. The auto-correlation thus provides a neat and simple method of separating the periodic and random portions of the time function.

7. Power Spectra. The spectrum corresponding to the periodic component of the auto-correlation is a line spectrum which differs from that of the unmodulated rectangular pulses only in that it contains smaller amounts of power in the higher-frequency components. This follows from the fact that the periodic auto-correlation component lacks the sharp corners of the auto-correlation of unmodulated pulses. The power removed from the high-frequency components as a result

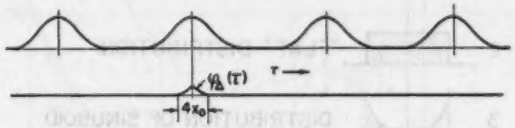

FIG. 3. Periodic and aperiodic components of auto-correlation shown in Fig. 2-b.

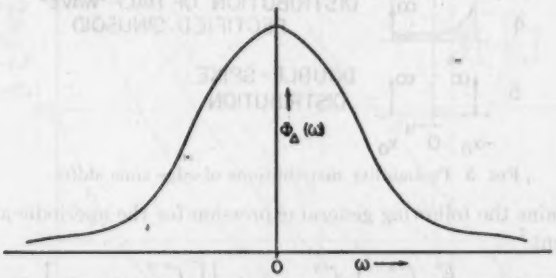

FIG. 4. General appearance of power spectrum of one-edge time-shift noise.

of the random time modulation of the pulse edges can be found in the continuous spectrum which corresponds to the aperiodic auto-correlation component. Attention will be focused on the power in the continuous spectrum, which produces audible random noise in the receiver output. The noise spectrum, expressed in watts per radian/sec, is obtained by taking the Fourier transform of the aperiodic auto-correlation component.

$$\Phi_\Delta(\omega) = (1/2\pi) \int_{-\infty}^{\infty} \varphi_\Delta(\tau) e^{-j\omega\tau}\, d\tau. \qquad (8)$$

Since the general shape of $\varphi_\Delta(\tau)$ is always as shown in Fig. 3, it follows that $\Phi_\Delta(\omega)$ always has the general appearance indicated in Fig. 4.

The significance of the power at negative frequencies is simply that the power density at any physical frequency is actually twice as large as indicated by $\Phi_\Delta(\omega)$. A factor of two should therefore multiply (8) when used to calculate noise power.

8. Quantitative Aspects of Auto-Correlations and Power Spectra of One-Edge Time-Shift Noise. The shape and magnitude of $\varphi_\Delta(\tau)$, and hence of $\Phi_\Delta(\omega)$, depend on the probability distribution of the pulse-edge time shifts. This, in turn, depends on the type of interference causing the shifts and may range all the way from distributions having standard deviations small compared to the maximum deviation (x_0), to those having standard deviations almost as large as the maximum deviation.

By deriving an expression for the rounded peaks of the auto-correlation (see Fig. 2-b) and subtracting it from the expression for the (unrounded) triangular

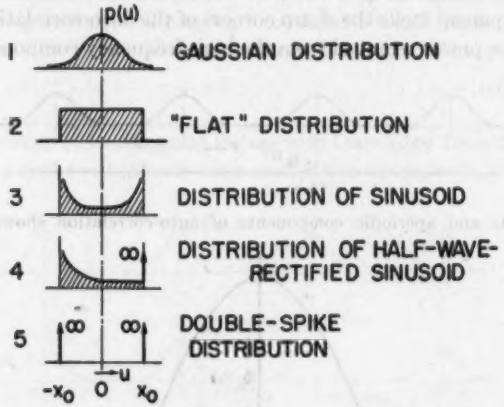

FIG. 5. Probability distributions of edge time shifts.

peak one obtains the following general expression for the aperiodic auto-correlation component.[2]

$$\varphi_\Delta(\tau) = \frac{E^2}{T} \int_{-x_0+\tau}^{x_0} \left[\int_x^{x_0} p(u)\, du\right]\left[\int_{x_0}^{x-\tau} p(u)\, du\right] dx. \qquad (9)$$

The result obtained from this formula for any particular probability distribution holds for $0 < \tau < 2x_0$. However, since $\varphi_\Delta(\tau)$ must be symmetrical about the origin, that portion of it lying in the region $-2x_0 < \tau < 0$ is given by the same result with τ replaced by its magnitude $|\tau|$. Outside of the interval $|\tau| < 2x_0$, the value of $\varphi_\Delta(\tau)$ is zero. Plots of $\varphi_\Delta(\tau)$ are shown in Fig. 6 for the five different probability distributions sketched and described in Fig. 5. The area under each curve of $\varphi_\Delta(\tau)$ is proportional to the dispersion (square of the standard deviation) of the corresponding probability distribution. This dispersion, in turn, is the same as the mean-square value of the periodic function which has an amplitude distribution identical to the probability distribution in question. For example, the periodic function corresponding to the sinusoidal distribution (dispersion = $x_0^2/2$) is obviously a sine wave, while a linear sawtooth wave corresponds to the "flat" distribution (dispersion = $x_0^2/3$), and a square wave corresponds to the double-spike distribution (dispersion = x_0^2). Inasmuch as $\varphi_\Delta(\tau)$ is symmetrical

[2] The derivation of (9) is outlined in the Appendix.

about $\tau = 0$, only the right-hand half ($\tau > 0$) is shown in Fig. 6; the other half is the mirror image. For each $\varphi_\Delta(\tau)$ the corresponding $\Phi_\Delta(\omega)$ is obtained by applying the Fourier transformation to $\varphi_\Delta(\tau)$. Since $\varphi_\Delta(\tau)$ has discontinuous derivatives at $|\tau| = 2x_0$ and zero, the method of translation and sectioning[3] is useful in finding the transform. The power-density spectra thus obtained for the five distributions of Fig. 5 are shown in normalized form in Fig. 7. An oscillatory term is present in all cases but is visible only for those distributions which favor the

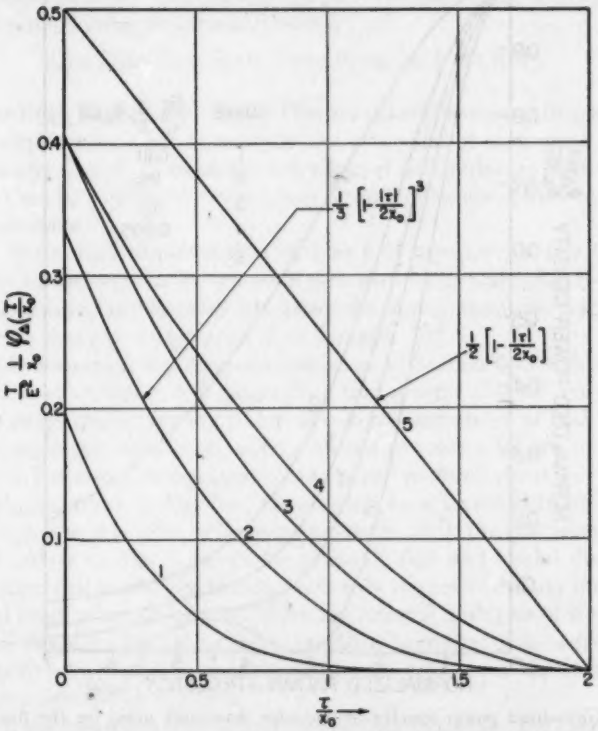

FIG. 6. Auto-correlations of one-edge time-shift noise for the five time-shift distributions given in Fig. 5.

extremes of the time-shift range. The power density at zero frequency is proportional to the area under the correlation curve and hence to the dispersion of the probability distribution. Conversely, the total noise power, that is, the area under the entire power-density curve, is proportional to $\varphi_\Delta(0)$.

9. Audio-Noise Power. The audio-frequency portion of the power spectra shown in Fig. 7 is confined to a narrow strip along the ordinate axis. For example, sup-

[3] M. F. Gardner and J. L. Barnes, "Transients in Linear Systems," Vol. 1, page 238.

pose that the peak time shift is one microsecond ($x_0 = 1$ μsec). The frequency corresponding to $x_0\omega = 1$ is then $1/2\pi$ or 0.16 Mc; consequently, if 16 kc is the highest audio frequency reproduced, then $x_0\omega = 0.1$ is the highest abscissa value of interest in the plots of Fig. 7. Thus it is readily seen that the power spectrum is substantially constant over the audio range ($0 < x_0\omega < 0.1$) for all five probability distributions. Ideally, a pulse modulation system may have an audio

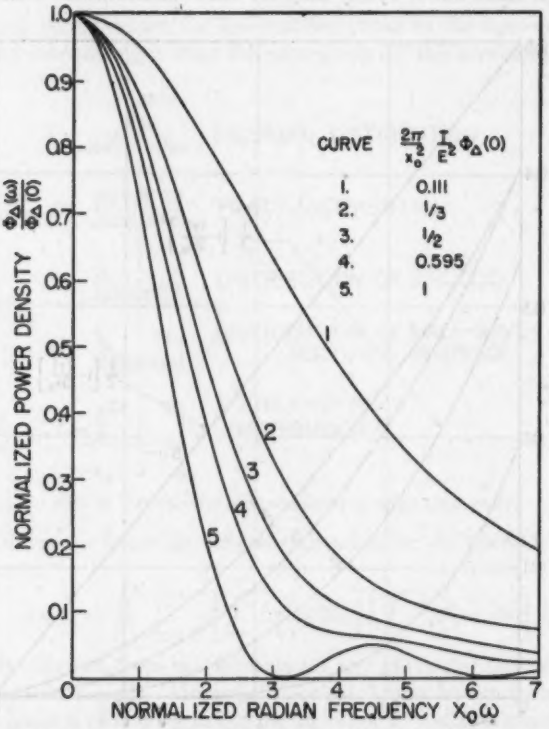

FIG. 7. Normalized power spectra of one-edge time-shift noise for the five time-shift distributions given in Fig. 5.

characteristic which is "flat" from zero to one-half the sampling frequency, $1/2T$. If such a characteristic (gain = 0 db) is assumed, the audio-noise power is given by $\int_0^{\pi/T} 2\,\Phi_\Delta(\omega)d\omega$, which is very nearly equal to $(2\pi/T)\,\Phi_\Delta(0)$; this, in turn, is found to equal $E^2(\overline{\Delta t}/T)^2$ where $\overline{\Delta t}$ is the standard deviation of the time-shift distribution.[4] Heuristic methods also lead to this latter result, but will not show the type and degree of approximation involved. The precise audio-noise power is

[4] If the time shifts are caused by interference from a sinusoidal carrier the "effective time shift" is given by $\overline{\Delta t} \approx 0.7\Delta t_{max}$.

slightly lower than that given in terms of the standard deviation, since pulse-time modulation does not generally involve true periodic sampling (because of the finite time deviations). True random periodic samples (such as amplitude-modulated impulses) have a perfectly flat power spectrum, so that the power in any band can be given exactly in terms of the standard deviation. It is of interest to note that the intuitive result, $E^2(\overline{\Delta t}/T)^2$ is correct only if the system has an ideal low-pass-filter characteristic. In general, the equivalent noise bandwidth of the audio system is approximately $1/3T$, so that the above expression for audio-noise power should be muliplied by two-thirds, yielding

$$\text{One-Edge Time-Shift Noise Power} \approx \tfrac{2}{3} E^2(\overline{\Delta t}/T)^2. \tag{10}$$

10. Two-Edge Time Shifts. Three different cases of two-edge time shifts are of practical significance: Case 1, in which the two edges of each pulse are shifted independently; Case 2, in which the two edges of each pulse are shifted by equal amounts; Case 3, in which the two edges of each pulse are shifted by equal but opposite amounts.

Case 1. If the analysis outlined in Section 6 is repeated for this Case, $\varphi_\Delta(\tau)$ is found to be the same as for one-edge time shifts, but multiplied by two. This means that $\varphi_\Delta(\omega)$ is also doubled in magnitude, that is, the noise is increased by 3 db but the shape of its spectrum is unchanged.

Case 2. Determining the auto-correlation as in Section 6, one finds that the auto-correlation around $\tau = 0$ differs from that around $\tau = nT$ not only in the curvature at the peaks (see Fig. 2) but also in the curvatures at both sides. As a result, the aperiodic component, $\varphi_\Delta(\tau)$, consists not merely of one lobe centered about $\tau = 0$ but also of two negative "side lobes" centered about $\tau = d$ and $-d$, respectively, as shown in Fig. 8-a. The center lobe is identical to that shown in Fig. 3, except that it is twice as large in amplitude, while the side lobes are identical to the curves of Fig. 3 except for algebraic sign and lateral displacement. Since the algebraic sum of the three areas is zero, the power density is also zero at $\omega = 0$ and at all other frequencies which are integral multiples of the reciprocal of the pulse duration. The power spectrum differs from that of one-edge time-shift noise (Fig. 6) only in that it is multiplied by the term $4 \sin^2 d\omega$. It may appear as in the example shown in Fig. 9.[5]

Case 3. If the two edges of each pulse are shifted oppositely, the aperiodic auto-correlation component, $\varphi_\Delta(\tau)$, is again found to consist of three lobes. However, the side lobes are positive and do not have the same shape as the center lobe, although their areas are the same as in Case 2, each being half the area of the center lobe (see Fig. 8-b). The power density at zero-frequency is four times that for one-edge time-shift noise; at low frequencies ($f < 1/d$) it is approximately the same as the spectrum of one-edge time-shift noise (Fig. 6) multiplied by the term $4 \cos^2 d\omega$. At higher frequencies, however, it departs radically from this approximation (see Fig. 8), as a result of the difference in shape between center lobe and side lobes of $\varphi_\Delta(\tau)$.

[5] Similar results are given by Z. Jelonek in "Noise Problems in Pulse Communication," Jour. IEE, pt. IIIA, vol. 94, no. 13, pp. 533–545; 1947.

While the total audio-noise power is independent of the average pulse duration or duty factor for one-edge and independent two-edge time shifts (Case 1), it has a first-order dependence on the duty factor in Case 2 and a second-order dependence in Case 3. Denoting the ratio of two-edge audio-noise power to one-edge

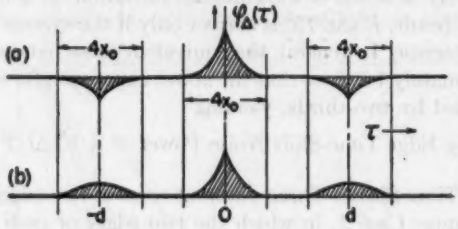

FIG. 8. Noise auto-correlations (aperiodic auto-correlation components) for two-edge time shifts: (a) Case 2; (b) Case 3.

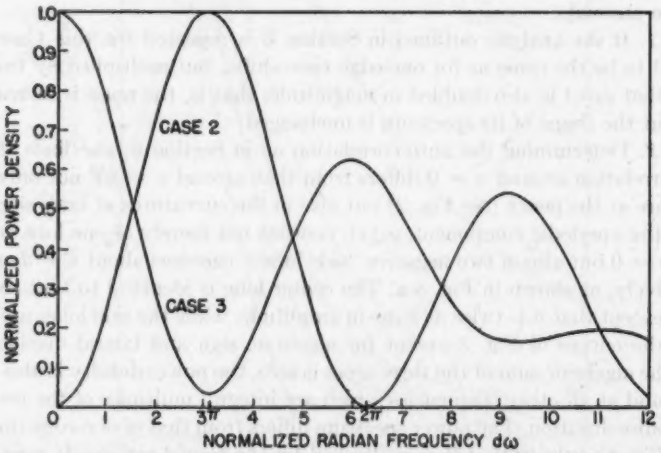

FIG. 9. Normalized power spectra of two-edge time-shift noise (Case 2 and Case 3) for "flat" probability distribution of time shifts.

audio-noise power by R_1, R_2, and R_3 for Cases 1, 2, and 3, respectively, one obtains

$$R_1 = 2, \quad R_2 \approx (\pi^2/3)D^2, \quad R_3 \approx 4 - (\pi^2/3)\,D^2. \tag{11}$$

11. Noise Caused by Randomly Missing Pulses. It was shown in Section 5 that the auto-correlation of the rectangular pulse train being considered is a series of triangles each of height $(E^2/T)\,d$. Random removal of a fraction F of the pulses reduces the height of the triangle centered about $\tau = 0$ by the factor $1 - F$, the same factor by which the total power is reduced. The height of all other triangles is reduced by the factor $(1 - F)^2$, since the overlap producing them is between two independent pulses both of which exist with a probability $1 - F$.

The auto-correlation can therefore again be split into a periodic component and an aperiodic component, the latter being a triangle of height $(E^2/T)\ d\ [F - F^2]$, extending from $\tau = -d$ to $\tau = d$. It is of exactly the same form as $\varphi_\Delta(\tau)$ for one-edge time shifts with a double-spike probability distribution (see Figs. 5 and 6), and can be obtained directly from (9) by giving to the two probability integrals the values F and $1 - F$, and letting $x_0 = \frac{1}{2}\ d$. The spectrum is therefore of the same form as that for the double-spike distribution in Fig. 7, with the abscissa variable $x_0\omega$ replaced by $\frac{1}{2}d\omega$ and with $\Phi_\Delta(0) = (1/2\pi)\ (E^2/T)\ d^2\ [F - F^2]$ watts per rad/sec. If the pulses are passed through an ideal low-pass filter, and the slight drop in power density from $\omega = 0$ to $\omega = \pi/T$ is neglected, the total audio noise power is $E^2D^2[F - F^2]$ watts. Assuming an equivalent noise bandwidth of $(1/3T)$ cps $(\frac{2}{3}(\pi/T)$ rad/sec) in a practical system, one obtains

Audio Noise Power due to Randomly Missing Pulses

$$\approx \tfrac{2}{3}E^2D^2[F - F^2] \text{ watts.} \quad (12)$$

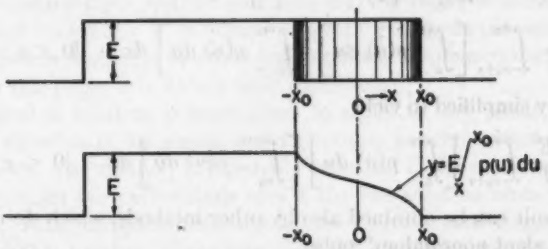

FIG. 10. Trailing pulse edge with random time shifts and "equivalent nonrandom" pulse edge.

12. Conclusions. The auto-correlation method of analysis has been used to provide the solution to a practical problem which is not readily solved rigorously by conventional methods. Besides giving the power and spectral distribution of the noise, the analysis also throws some light on the nature of random time modulation. For example, it shows that the time-shift noise is generated at the expense of the high-frequency components of the pulse-train spectrum and permits exact calculation of this exchange of power. This method should therefore prove a powerful tool in various modulation problems, keeping pace with the increasing application of statistical theory to communications.

Acknowledgment. The author is indebted to Professors Y. W. Lee, J. B. Wiesner, and Mr. T. P. Cheatham, Jr., for their valuable suggestions, as well as to the Research Laboratory of Electronics for its services and facilities.

Appendix. The derivation of (9) can be outlined as follows. A large number of pulses whose trailing edges have random time shifts with probability distribution $p(u)$ are replaceable by a single equivalent nonrandom pulse as shown in Fig. 10. The auto-correlation for $-2x_0 < \tau + nT < 2x_0$ $(n \neq 0)$ is found by

multiplying together two of these "equivalent" pulses displaced by an amount τ. This leads to the following equation, which describes the curved peak of any of the auto-correlation lobes (except that around $\tau = 0$), moved to the τ origin (Fig. 2-b).

$$\varphi_{n\neq 0}^{(\tau)} = \frac{E^2}{T}\left\{d - x_0 - \tau + \int_{x_0}^{-x_0+\tau}\left[\int_x^{x_0} p(u)\,du\right]dx + \int_{-x_0+\tau}^{x_0}\left[\int_x^{x_0} p(u)\,du\right]\left[\int_{x-\tau}^{x_0} p(u)\,du\right]dx\right\} \qquad [0 < \tau < 2x_0]. \quad (13)$$

The auto-correlation for τ near zero is triangular and is given by

$$\varphi_{n=0}^{(\tau)} = (E^2/T)\,(d - \tau) \qquad [0 < \tau < d]. \quad (14)$$

Subtraction of (13) from (14) gives the aperiodic auto-correlation component.

$$\varphi_\Delta(\tau) = \frac{E^2}{T}\left\{x_0 - \int_{x_0}^{-x_0+\tau}\left[\int_x^{x_0} p(u)\,du\right]dx - \int_{-x_0+\tau}^{x_0}\left[\int_x^{x_0} p(u)\,du\right]\left[\int_{x-\tau}^{x_0} p(u)\,du\right]dx\right\} \qquad [0 < \tau < 2x_0]. \quad (15)$$

This is readily simplified to yield

$$\varphi_\Delta(\tau) = \frac{E^2}{T}\int_{-x_0+\tau}^{x_0}\left[\int_x^{x_0} p(u)\,du\right]\left[\int_{x_0}^{x-\tau} p(u)\,du\right]dx \qquad [0 < \tau < 2x_0]. \quad (9)$$

The same result can be obtained also by other methods, which do not make use of the "equivalent nonrandom" pulse.

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A NEW FORM OF SOLUTION OF HERMITE'S EQUATION[1]

By W. A. Mersman

Introduction. A modified form of Hermite's differential equation has recently been encountered by Chapman and Rubesin (reference 1) in a study of the laminar boundary layer of a compressible fluid flowing over a heated flat plate. The equation is

$$\frac{d^2u}{dz^2} + 2z\frac{du}{dz} - 4nu = 0, \qquad n = 0, 1, 2, 3, \cdots,$$

in which u is a dimensionless coefficient occurring in a series expansion for the temperature, and z is a distorted dimensionless coordinate representing distance normal to the plate. The surface of the plate is approximately at $z = 0$, and the free stream outside the boundary layer is at $z = \infty$.

In the boundary-layer problem solutions $u(z)$ are required which are valid for positive z and vanish at $z = \infty$. Since classical methods do not seem to lead to a solution which is tractable to numerical methods of computation, the primary objective of this paper is to obtain such a solution.

The method of solution is generalized to apply to any linear, homogeneous differential equation of the second order to obtain a second solution when one is already known. The method is especially powerful if the coefficients of the differential equation are polynomials and if the one solution already known is a polynomial. Thus, complete solutions of the Hermite and Laguerre equations are obtained that involve only polynomials and error function integrals or exponential integrals, both of which have been extensively tabulated. In the case of the Bessel and Legendre equations the method leads to the well-known functions of the second kind in a direct manner.

The general linear equation of the second order. Consider the equation

$$p(z)\frac{d^2u}{dz^2} + q(z)\frac{du}{dz} + r(z)u = 0, \tag{2.1}$$

and let $y(z)$ be a known solution. The classical method of obtaining a second solution is to make the substitution (reference 2, p. 101)

$$u = yv,$$

which leads to the following equation for v:

$$\frac{d^2v}{dz^2} + \left(\frac{q}{p} + \frac{2}{y}\frac{dy}{dz}\right)\frac{dv}{dz} = 0.$$

This is immediately integrable to give

$$v = \int \frac{e^Q}{y^2}\,dz, \qquad Q = -\int \frac{q}{p}\,dz.$$

[1] Presented to the American Mathematical Society on April 30, 1949.

Unfortunately, this integral representation of v is usually difficult to evaluate analytically or numerically. For the classical equations of physical interest $v(z)$ takes the following forms:

$$\text{Hermite's equation:} \qquad v = \int \frac{e^{z^2}}{[H_n(z)]^2}\, dz,$$

$H_n(z)$ being the Hermite polynomial of degree n.

$$\text{Modified Hermite equation:} \qquad v = \int \frac{e^{-z^2}}{[H_{2n}(iz)]^2}\, dz,$$

$$\text{Laguerre's equation:} \qquad v = \int \frac{e^z}{z[L_n(z)]^2}\, dz,$$

$L_n(z)$ being the Laguerre polynomial of degree n:

$$\text{Bessel's equation:} \qquad v = \int \frac{dz}{z[J_n(z)]^2},$$

$J_n(z)$ being the Bessel function of the first kind of order n.

$$\text{Legendre's equation:} \qquad v = \int \frac{dz}{(1 - z^2)[P_n(z)]^2},$$

$P_n(z)$ being the Legendre polynomial of degree n.

In every case it is evident that the integral representation of $v(z)$ is extremely unwieldy, both analytically and numerically. In the cases of Bessel's and Legendre's equations alternative forms of the second solution are well known. One method of obtaining them (reference 2, pp. 170 and 181) can be generalized as follows: if $y(z)$ is a known solution of equation (2.1), the substitution

$$u = yv + w \tag{2.2}$$

reduces that equation to

$$p\frac{d^2w}{dz^2} + q\frac{dw}{dz} + rw = -y\left(p\frac{d^2v}{dz^2} + q\frac{dv}{dz}\right) - 2p\frac{dv}{dz}\frac{dy}{dz}.$$

The coefficient of y in the right member can be made to vanish by choosing v to be any solution of the equation $p\, d^2v/dz^2 + q\, dv/dz = 0$. A suitable choice is

$$v = \int e^Q\, dz, \qquad Q = -\int (q/p)\, dz. \tag{2.3}$$

A second solution of equation (2.1) is then given by equation (2.2), provided that w is any solution of

$$p\frac{d^2w}{dz^2} + q\frac{dw}{dz} + rw = -2pe^Q\frac{dy}{dz} \tag{2.4}$$

where Q is defined by equation (2.3), and y is any known solution of the original equation (2.1).

Hermite's equation of integral order. Hermite's equation of integral order, n, is (reference 3, p. 59)

$$\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + 2nu = 0, \qquad n = 0, 1, 2, 3, \cdots. \tag{3.1}$$

Using the notation of section 2, one solution of this equation is the Hermite polynomial of degree n,

$$y = H_n(z) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!}(2z)^{n-2k}, \qquad n = 0, 1, 2, 3, \cdots, \tag{3.2}$$

where $[n/2]$ is the greatest integer not greater than $n/2$. The coefficients of the equation are

$$p = 1, \qquad q = -2z, \qquad r = 2n.$$

Hence, from equation (2.3),

$$Q = z^2, \qquad v = \int e^{z^2}\,dz,$$

and a second solution of Hermite's equation (3.1) is

$$u = H_n(z)\int e^{z^2}\,dz + w, \tag{3.3}$$

where w is any solution of the equation

$$\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + 2nw = -2e^{z^2}\frac{dH_n}{dz}.$$

It is convenient to make the substitution

$$w = G_n(z)e^{z^2}.$$

Then $G_n(z)$ is to be any solution of the equation

$$\frac{d^2G_n}{dz^2} + 2z\frac{dG_n}{dz} + 2(n+1)G_n = -2\frac{dH_n}{dz}, \qquad n = 0, 1, 2, 3, \cdots.$$

The standard method of expansion in a power series leads to the solution

$$G_0(z) = 0$$

$$G_n(z) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^{k+1}\frac{(n-k-1)!}{(n-2k-1)!}\sum_{j=0}^{k}\frac{n!}{j!(n-j)!}(2z)^{n-2k-1}$$

$$n = 1, 2, 3, \cdots. \tag{3.4}$$

Hence, the general solution of Hermite's equation of integral order, n, equation (3.1) is

$$u = AH_n(z) + B\left[H_n(z)\int_0^z e^{t^2}\,dt + G_n(z)e^{z^2}\right] \tag{3.5}$$

where $G_n(z)$ is given by equation (3.4), $H_n(z)$ by equation (3.2), and A and B are arbitrary constants.

This form of solution is well adapted to numerical computation, since the exponential function and integral are tabulated (reference 4).

The first few polynomials, $G_n(z)$ are: $G_0 = 0$, $G_1 = -1$, $G_2 = -2z$, $G_3 = -4z^2 + 4$, $G_4 = -8z^3 + 20z$, $G_5 = -16z^4 + 72z^2 - 32$, $G_6 = -32z^5 + 224z^3 - 264z$, $G_7 = -64z^6 + 640z^4 - 1392z^2 + 384$.

The modified Hermite equation. Replacing z by iz and n by $2n$ in equation (3.1) yields the modified Hermite equation mentioned in the introduction:

$$\frac{d^2u}{dz^2} + 2z\frac{du}{dz} - 4nu = 0, \qquad n = 0, 1, 2, 3, \cdots. \tag{4.1}$$

The same transformations applied to equation (3.5) give the solution of equation (4.1):

$$u(z) = AH_{2n}(iz) + B\left[iH_{2n}(iz)\int_0^z e^{-t^2}\,dt + G_{2n}(iz)e^{-z^2}\right].$$

Equation (3.4) shows that $G_{2n}(iz)$ is pure imaginary. If, then, $G_{2n}(iz)$ is set equal to $iK_{2n}(z)$, and i is incorporated into the arbitrary constant, B, the solution of equation (4.1) can written

$$u(z) = AH_{2n}(iz) + B\left[H_{2n}(iz)\int_0^z e^{-t^2}\,dt + K_{2n}(z)e^{-z^2}\right] \tag{4.2}$$

where

$$H_{2n}(iz) = (-1)^n\sum_{k=0}^{n}\frac{(2n)!}{k!(2n-2k)!}(2z)^{2n-2k}, \qquad n = 0, 1, 2, 3, \cdots. \tag{4.3}$$

$$K_0 = 0, \qquad K_{2n}(z) = (-1)^n\sum_{k=0}^{n-1}\frac{(2n-k-1)!}{(2n-2k-1)!}\sum_{j=0}^{k}\frac{(2n)!}{j!(2n-j)!}(2z)^{2n-2k-1},$$

$$n = 1, 2, 3, \cdots. \tag{4.4}$$

Again the solution involves tabulated exponenial functions and integrals.

It is convenient to express the arbitrary constants A and B in equation (4.2) in terms of the initial values. This is easily done by noting that for n = 0, 1, 2, 3, $\cdots$,

$$H_{2n}(0) = (-1)^n(2n)!/n!$$

$$[dH_{2n}(iz)/dz]_{z=0} = 0, \qquad K_{2n}(0) = 0,$$

$$\left(\frac{dK_{2n}}{dz}\right)_{z=0} = (-1)^n n!\,2\sum_{j=0}^{n-1}\frac{(2n)!}{j!(2n-j)!} = (-1)^n\left[4^n n! - \frac{(2n)!}{n!}\right].$$

Hence, from equation (4.2),

$$u_0 \equiv u(0) = A(-1)^n(2n)!/n!, \qquad n = 0, 1, 2, 3, \cdots,$$

$$u_0' \equiv (du/dz)_{z=0} = B(-1)^n 4^n n!, \qquad n = 0, 1, 2, 3, \cdots,$$

and the solution (equation (4.2)) of the modified Hermite equation (4.1) can be written in the form

$$u(z) = \frac{(-1)^n n!}{(2n)!}\, u_0 H_{2n}(iz)$$

$$+ \frac{(-1)^n}{4^n n!}\, u_0' \left[H_{2n}(iz) \int_0^z e^{-t^2}\, dt + K_{2n}(z) e^{-z^2} \right], \qquad n = 0, 1, 2, 3, \cdots, \quad (4.5)$$

where $H_{2n}(iz)$ and $K_{2n}(z)$ are given by equations (4.3) and (4.4). Noting that

$$\int_0^z e^{-t^2}\, dt = (\pi^{1/2}/2) - \int_z^\infty e^{-t^2}\, dt,$$

the solution (equation (4.5)) can be put in the form

$$u(z) = \frac{(-1)^n}{4^n n!}\, H_{2n}(iz) \left[\frac{u_0' \pi^{1/2}}{2} + \frac{4^n (n!)^2}{(2n)!}\, u_0 \right]$$

$$- \frac{(-1)^n}{4^n n!}\, u_0' \left[H_{2n}(iz) \int_z^\infty e^{-t^2}\, dt - K_{2n}(z) e^{-z^2} \right].$$

Since $H_{2n}(iz)$ and $K_{2n}(z)$ are polynomials, the last two terms vanish at $z = \infty$, z real.

In the boundary-layer problem mentioned in the introduction it is required to find a solution which vanishes at $z = \infty$. For a solution in the form obtained above this condition imposes a relation between u_0 and u_0' which is

$$u_0' = -\frac{2^{2n+1}(n!)^2}{\pi^{1/2}(2n)!}\, u_0 .$$

When this is satisfied, the solution can be written

$$u(z) = \frac{(-1)^n n!}{(2n)!}\, u_0 \left[H_{2n}(iz)\, \frac{2}{\pi^{1/2}} \int_z^\infty e^{-t^2}\, dt - \frac{2}{\pi^{1/2}}\, K_{2n}(z) e^{-z^2} \right]. \quad (4.7)$$

The function $u(z)$ defined by equation (4.7) is a solution of the modified Hermite equation (4.1). Since it vanishes at $z = \infty$ and is well-behaved at $z = 0$, it is a suitable function to be taken as the standard second solution. Then the general solution of equation (4.1) can be written in the form

$$u(z) = AH_{2n}(iz) + BF_n(z) \quad (4.8)$$

where

$$F_n(z) = \frac{(-1)^n n!}{(2n)!} \left[H_{2n}(iz)\, \frac{2}{\pi^{1/2}} \int_z^\infty e^{-t^2}\, dt - \frac{2}{\pi^{1/2}}\, K_{2n}(z) e^{-z^2} \right]. \quad (4.9)$$

Laguerre's equation of integral order. Laguerre's equation of integral order is (reference 3, p. 58)

$$z \frac{d^2u}{dz^2} + (1 - z) \frac{du}{dz} + nu = 0, \qquad n = 0, 1, 2, 3, \cdots. \quad (5.1)$$

Using the notation of section 2, one solution is the Laguerre polynomial of degree n,

$$y = L_n(z) = \sum_{k=0}^{n} \left(\frac{n!}{k!}\right)^2 \frac{(-z)^k}{(n-k)!}. \tag{5.2}$$

The coefficients of the differential equation are

$$p = z, \qquad q = 1 - z, \qquad r = n.$$

Hence, by equation (2.3),

$$Q = \int (1 - z^{-1})\, dz = z - \ln z,$$

$$v = \int e^z z^{-1}\, dz,$$

and, by equations (2.2) and (2.4), a second solution of equation (5.1) is given by

$$u = L_n(z) \int e^z z^{-1}\, dz + w \tag{5.3}$$

where w is any solution of the equation

$$z \frac{d^2w}{dz^2} + (1 - z) \frac{dw}{dz} + nw = -2e^z \frac{dL_n}{dz}.$$

It is convenient to make the substitution

$$w = e^z M_n(z). \tag{5.4}$$

Then $M_n(z)$ is to be any solution of the equation

$$z \frac{d^2M_n}{dz^2} + (z + 1) \frac{dM_n}{dz} + (n + 1)M_n = -2 \frac{dL_n}{dz}.$$

The standard method of expansion in series gives the solution

$$M_0 = 0$$

$$M_n(z) = \sum_{k=0}^{n-1} (-1)^k \frac{2(n + k)!(n!)^2}{(k!)^2} \sum_{j=k}^{n-1} \frac{1}{(j + 1)(n + j + 1)!(n - j - 1)!} z^k,$$

$$n = 1, 2, 3, \cdots. \tag{5.6}$$

Thus, from equations (5.2), (5.3), (5.4), and (5.6), the complete solution of Laguerre's equation of integral order, n, (equation (5.1)) is

$$u = AL_n(z) + B\left[L_n(z) \int_{\infty}^{z} e^t t^{-1}\, dt + e^z M_n(z)\right], \quad n = 0, 1, 2, 3, \cdots, \tag{5.7}$$

where A and B are arbitrary constants, $L_n(z)$ is the Laguerre polynomial, given by equation (5.2), and $M_n(z)$ is the polynomial given by equation (5.6). The exponential function and integral have been tabulated (reference 4).

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BEST INTERPOLATION FORMULAS

By Leroy F. Meyers and Arthur Sard[1]

1. Introduction. Suppose that tabular values $x(0), x(1), x(2), \cdots$ of a function $x = x(t)$ are given. Suppose that we wish to approximate $x(u)$, u real and fixed, by an expression of the form

$$A = a_0x(0) + a_1x(1) + \cdots + a_mx(m). \tag{1}$$

What value of m shall we use and what coefficients $a_0 = a_0(u)$, $a_1 = a_1(u)$, $\cdots$, $a_m = a_m(u)$?

Conventional practice has been to specify that the approximation be exact: $A = x(u)$ whenever $x(t)$ is a polynomial in t of degree n and to use in (1) the $n + 1$ entries of the set $x(0), x(1), \cdots$ nearest u. Thus, if $u \leqq (n + 1)/2$, then $m = n$; if $(n + 1)/2 \leqq u \leqq (n + 3)/2$, then $m = n + 1$ and $a_0 = 0$; and so on. We shall see that the conventional practice is not always best.

If the approximation (1) is exact whenever $x(t)$ is a polynomial of degree n, then there is a kernel function $k = k(t, u)$ such that

$$R[x] = x(u) - A = \int_K x^{(n+1)}(t)k(t, u)\, dt \tag{2}$$

whenever $x(t)$ has a continuous $(n+1)$th derivative $x^{(n+1)}(t)$, where K is the smallest interval containing u and those of the values $0, 1, \cdots, m$ for which the corresponding coefficients $a_0, a_1, \cdots, a_m$ are not zero. Indeed $k(t, u)$, for each u, is a broken polynomial in t consisting of at most $m + 3$ arcs; k is defined [2] simply in terms of $R[x]$:

$$k(t', u) = R[\psi_{t'}] = -R[\varphi_{t'}], \tag{3}$$

where

$$\psi_{t'} = \psi_{t'}(t) = \begin{cases} 0 \\ (t - t')^n/n! \end{cases}, \qquad \varphi_{t'} = \varphi_{t'}(t) = \begin{cases} (t - t')^n/n! & \text{if } t \leqq t', \\ 0 & \text{if } t > t'. \end{cases}$$

Let $\mathfrak{C}_{m,n,u}$ be the class of all approximations (1) which are exact for degree n, where m, n, u are fixed. The class $\mathfrak{C}_{m,n,u}$ is not empty if $m \geqq n$; it consists of one element (the conventional formula) if $m = n$ or if $m < n$ and u is one of the integers $0, 1, \cdots, m$; in all other cases $\mathfrak{C}_{m,n,u}$ is empty. (See §4).

Schwarz's inequality applied to (2) implies that

$$|R[x]| \leqq M\left[\int_K x^{(n+1)}(t)^2\, dt/|K|\right]^{1/2}, \tag{4}$$

where

$$M = M(u) = \left[|K| \int_K k(t, u)^2\, dt\right]^{1/2}$$

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is the *modulus* of the approximation A and $|K|$ is the length of K. Observe that the multiplier of M in (4) is the root-mean-square of $x^{(n+1)}$ on K. We suppose that (4) is to be used to appraise $R[x]$ in a situation in which we have no knowledge as to which sets K are such that the root-mean-square of $x^{(n+1)}$ on K is small or large. In such a situation it is natural to say that that $A \in \mathfrak{A}_{m,n,u}$ is *best* which minimizes the modulus M. We adopt this criterion [1]. Let $A_{m,n,u}$ be the best approximation in $\mathfrak{A}_{m,n,u}$, or one of the best if there are several.

Throughout our discussion n is a given, non-negative integer; n determines the order of the derivative in terms of which we choose to appraise $R[x]$.

The formula $A_{m,n,u}$ may be the same as $A_{m+1,n,u}$; this will be the case when the coefficient a_{m+1} of $A_{m+1,n,u}$ vanishes.

Observe that there is no initial restriction whatever on the nature of the coefficients $a_0, \cdots, a_m$ as functions of u. It turns out that these coefficients in $A_{m,n,u}$ are broken polynomials in u of degree $2n + 1$. It turns out also that the best formulas are interpolatory: $A_{m,n,u} = x(u)$ if u is an integer and $0 \leqq u \leqq m$.

The present paper is a report of results already established and of surmises as yet unresolved.

We have proved the following.

For $n = 0$, all m, the conventional practice is best. Thus $A_{m,0,u} = x(i_0)$ where i_0 is an integer in the range $0 \leqq i_0 \leqq m$ nearest u. In this case $M^2 = (u - i_0)^2$.

For $n = 1$, $m = 1, 2, 3, 4$, the conventional practice is best:

$$A_{m,1,u} = (i_0 + 1 - u)\, x(i_0) + (u - i_0)x(i_0 + 1),$$

where i_0 is an integer and $0 \leqq i_0 \leqq u \leqq i_0 + 1 \leqq m$, or $u < 0$, $i_0 = 0$, or $u > m$, $i_0 = m - 1$. In all these cases $M^2 = (i_0 + 1 - u)^2\,(u - i_0)^2/3$. We surmise that these assertions are true for all $m \geqq 1$.

For $n = 2$, $m = 3, 4, 5$, the conventional practice is not best. (For $n = 2$, $m = 2$, the conventional practice is best since $\mathfrak{A}_{2,2,u}$ consists of but a single element.) The approximation $A_{5,2,u}$, depending on the value of u, involves 1, 3, 4, or 5 but never 2 or 6 consecutive tabular values. The approximations $A_{2,2,u}$, $A_{3,2,u}$, $A_{4,2,u}$ and $A_{5,2,u}$ are, in effect, given in §3 for all u. We surmise that $A_{m,2,u} = A_{4,2,u}$ for $m \geqq 4$.

Once the best formulas have been determined their use is direct. If one were to interpolate extensively one could tabulate the coefficients $a_0, \cdots, a_m$ for different u.

We discuss the derivation of the formulas in §4 and their transformation under a change of origin and scale in §5.

Remark. In this paper we minimize the modulus $M(u)$ for each u. A different procedure, which might be called *best graduation*, is the following. Let $\mathfrak{A}^*_{m,n,q}$ be the class of approximations (1) which are exact for all u whenever $x(t)$ is a polynomial of degree n and in which the coefficients $a_i = a_i(u)$ are polynomials in u of degree q, where $m \geqq n$, $q \geqq n \geqq 0$. For each A in $\mathfrak{A}^*_{m,n,q}$ there is a modulus function $M(u)$. One might say that that $A \in \mathfrak{A}^*_{m,n,q}$ is best which minimizes $\int_0^m M^2(u)\, du$ or, alternatively, sup $M^2(u)$, $0 \leqq u \leqq m$ [1, last paragraph].

2. The preparatory formulas. For an approximation $A \in \mathfrak{A}_{m,n,u}$, we define J as follows:

$$J = \int_{\min(u,0)}^{\max(u,m)} k(t, u)^2 \, dt. \tag{5}$$

There is a unique approximation $A \in \mathfrak{A}_{m,n,u}$ which minimizes J. (See §4.) We denote this approximation by $B_{m,n,u}$. The determination of the approximation $A_{m,n,u}$ involves the prior determination of a *finite* number of preparatory approximations, viz.

$$B_{p,n,u-j}, \qquad j = 0, 1, \cdots, m \quad p; \qquad p = n, n+1, \cdots, m, \tag{6}$$

and the comparison of the moduli of these approximations. The arguments $u - j$ enter here because of the possibility of shifting. For example, $B_{3,2,u-1}$ applied to $x(1), x(2), x(3), x(4)$ is an approximation in $\mathfrak{A}_{4,2,u}$. In the case of $A_{4,2,u}$ the complete set (6) consists of $B_{4,2,u}$, $B_{3,2,u}$, $B_{2,2,u}$, $B_{3,2,u-1}$, $B_{2,2,u-1}$, $B_{2,2,u-2}$.

We have proved that for $n = 2$, $m \leqq 5$, $0 \leqq u \leqq m$, extrapolation is inferior to interpolation; that is, the approximations (6), if any, with $u - j$ negative never carry the least modulus. Also, that for $n = 2$, $m \leqq 5$, $u < 0$ or $u > m$, the conventional practice is best.

We surmise that extrapolation is always inferior to interpolation, when $0 \leqq u \leqq m$. We surmise also that when extrapolation is necessary: $u < 0$ or $u > m$, the conventional practice is always best.

The approximations $B_{m,n,u}$ are interesting in themselves, as there may be situations in which one chooses to use many tabular values in order to distribute tabulated errors, for example.

We denote the modulus of $B_{m,n,u}$ by $M_{m,n,u}$.

3. Best formulas, $n = 2$. Throughout this section, $n = 2$. Table 1 gives the values of

$$\mu = M_{m,2,u}^2/\theta(u),$$

where

$$\theta(u) = (u - [u])^2 \, (1 - u + [u])^2/120 \tag{7}$$

and $[u]$ is the largest integer less than or equal to u. Since $\theta(u)$ depends on $u - [u]$ only, θ is the same for all the $u - j$ in (6). Accordingly, a comparison of the values of μ for (6) is equivalent to a comparison of the moduli.

Table 1 is based on the expressions for $M_{m,2,u}^2$ given below.

The entries in bold face type correspond to the approximation $A_{4,2,u} = A_{5,2,u}$. Observe that the approximation $B_{3,2,u}$, shifted if necessary, is either best or close to best for all interior u. None of the entries for $u = 2.4, 2.5$ is in bold face type. This is because the entries for $u = 1.4, 1.5$ and $m = 3$ are smaller.

The formula $B_{m,n,u}$ or $A_{m,n,u}$, $u \geqq m/2$, can be derived from its counterpart for $u \leqq m/2$: One replaces a_i by a_{m-i}, $i = 0, 1, \cdots, m$, and u by $m - u$, both in the coefficients and in the modulus. Accordingly we need consider $u \leqq m/2$ only.

Table 1 implies that

$$B_{2,2,u} = A_{2,2,u} = A_{3,2,u} = A_{4,2,u} = A_{5,2,u}, \qquad 0 \leqq u \leqq .5,$$

$$B_{3,2,u} = A_{3,2,u} = A_{4,2,u} = A_{5,2,u}, \qquad .6 \leqq u \leqq 1.6,$$

$$B_{4,2,u} = A_{4,2,u} = A_{5,2,u}, \qquad 1.7 \leqq u \leqq 2.3.$$

TABLE 1

Values of μ

u \ m	2	3	4	5
0	**20.00**	22.90	29.13	36.09
.1	**18.58**	20.90	26.49	32.80
.2	**17.14**	18.88	23.83	29.49
.3	**15.67**	16.87	21.20	26.21
.4	**14.21**	14.90	18.62	22.99
.5	**12.75**	12.98	16.12	19.88
.6	11.31	**11.14**	13.73	16.91
.7	9.91	**9.39**	11.48	14.11
.8	8.54	**7.76**	9.39	11.52
.9	7.24	**6.26**	7.48	9.16
1.0	6.00	**4.90**	5.78	7.05
1.1	7.24	**5.53**	6.41	7.80
1.2	8.54	**6.05**	6.88	8.34
1.3	9.91	**6.43**	7.16	8.64
1.4	11.31	**6.67**	7.24	8.68
1.5	12.75	**6.75**	7.12	8.49
1.6	14.21	**6.67**	6.81	8.06
1.7	15.67	6.43	**5.32**	7.43
1.8	17.14	6.05	**5.70**	6.63
1.9	18.58	5.53	**4.96**	5.70
2.0	20.00	4.90	**4.15**	4.71
2.1		6.26	**4.96**	5.53
2.2		7.76	**5.70**	6.23
2.3		9.39	**6.32**	6.76
2.4		11.14	6.81	7.10
2.5		12.98	7.12	7.21

For $2.4 \leqq u$, the best formulas are obtained by shifting. For example, for $2.4 \leqq u \leqq 3.4$, $A_{5,2,u} = A_{4,2,u}$ = the approximation $B_{3,2,u-1}$ operating on $x(1)$, $x(2)$, $x(3)$, $x(4)$.

Central Interpolation. Suppose that one wishes to interpolate in a table, away from the ends, and that one wishes to use six ordinates or fewer, with $n = 2$. Table 1 implies that, given u, one should translate, if necessary, so as to use the u given in Table 2, depending on the value of $u - [u]$.

We now list $B_{m,2,u}$, $0 \leqq u \leqq m/2$, $m = 2, 3, 4$; and $M^2_{m,2,u}$, $u \leqq m/2$, $m = 2, 3,$

4, 5. We write x_i for $x(i)$. The function θ is defined in (7). *The approximations* $B_{m,2,u}$, $u \leqq m/2$, $m \leqq 5$, *which are omitted are never best.*

$$2B_{2,2,u} = (u-1)(u-2)x_0 - 2u(u-2)x_1 + u(u-1)x_2, \qquad \text{all } u;$$

$$M^2_{2,2,u} = 2\theta(u)(u^3 - u^2 - 7u + 10), \qquad 0 \leqq u \leqq 1,$$

$$M^2_{2,2,u} = (2-u)u^2(-6u^3 + 23u^2 - 27u + 10)/120, \qquad u \leqq 0.$$

$$\begin{aligned} 132B_{3,2,u} = {} & (u-1)(-2u^4 - 2u^3 - 2u^2 + 91u - 132)x_0 \\ & + 3u(2u^4 - 71u + 113)x_1 + 3u(u-1)(-2u^3 - 2u^2 - 2u + 47)x_2 \\ & + u(u-1)(2u^3 + 2u^2 + 2u - 25)x_3, \qquad 0 \leqq u \leqq 1, \end{aligned}$$

TABLE 2

$u - [u]$	Translated u	Value of m such that $A_{4,2,u} = B_{m,2,u}$	Corresponding μ
.1	2.1	4	4.96
.2	2.2	4	5.70
.3	2.3	4	6.32
.4	1.4	3	6.67
.5	1.5	3	6.75
.6	1.6	3	6.67
.7	1.7	4	6.32
.8	1.8	4	5.70
.9	1.9	4	4.96

$$\begin{aligned} 132B_{3,2,u} = {} & (u-1)(u-2)(4u^3 - 18u^2 - 2u + 63)x_0 \\ & + 3(u-2)(-4u^4 + 22u^3 - 16u^2 - 43u - 3)x_1 \\ & + 3(u-1)(4u^4 - 26u^3 + 34u^2 + 23u + 6)x_2 \\ & + (u-1)(u-2)(-4u^3 + 18u^2 + 2u + 3)x_3, \qquad 1 \leqq u \leqq 2; \end{aligned}$$

$$M^2_{3,2,u} = \theta(u)(-4u^6 + 8u^5 - 12u^4 + 356u^3 - 168u^2 - 1748u + 2015)/88, \qquad 0 \leqq u \leqq 1,$$

$$M^2_{3,2,u} = \theta(u)(-16u^6 + 144u^5 - 308u^4 - 312u^3 + 1208u^2 - 276u - 9)/88, \qquad 1 \leqq u \leqq 2,$$

$$M^2_{3,2,u} = (3-u)u^2(-1584u^3 + 5343u^2 - 5778u + 2015)/31680, \qquad u \leqq 0.$$

$$\begin{aligned} 1840B_{4,2,u} = {} & (u-1)(-33u^4 - 33u^3 - 33u^2 + 1326u - 1840)x_0 \\ & + 4u(28u^4 - 829u + 1261)x_1 + 46u(u-1)(-3u^3 - 3u^2 - 3u + 56)x_2 \\ & + 4u(u-1)(18u^3 + 18u^2 + 18u - 211)x_3 \\ & + u(u-1)(-13u^3 - 13u^2 - 13u + 146)x_4, \qquad 0 \leqq u \leqq 1, \end{aligned}$$

$$1840B_{4,2,u} = (u-1)(u-2)(79u^3 - 323u^2 - 7u + 864)x_0$$
$$+ 4(u-2)(-74u^4 + 362u^3 - 296u^2 - 401u - 51)x_1$$
$$+ 46(u-1)(9u^4 - 51u^3 + 69u^2 + 8u + 12)x_2$$
$$+ 4(u-1)(u-2)(-64u^3 + 218u^2 - 38u + 41)x_3$$
$$+ (u-1)(u-2)(59u^3 - 183u^2 + 53u - 36)x_4, \qquad 1 \leqq u \leqq 2;$$

$$M^2_{4,2,u} = \theta(u)(-33u^6 - 66u^5 - 99u^4 + 2586u^3 - 1061u^2 - 12068u + 13398)/460, \qquad 0 \leqq u \leqq 1,$$

$$M^2_{4,2,u} = \theta(u)(-217u^6 + 1658u^5 - 3351u^4 - 1458u^3 + 7935u^2 - 1808u - 102)/460, \qquad 1 \leqq u \leqq 2$$

$$M^2_{4,2,u} = (4-u)u^2(-11040u^3 + 36473u^2 - 38864u + 13398)/220800, \qquad u \leqq 0.$$

$$M^2_{5,2,u} = \theta(u)(-2944u^6 - 5888u^5 - 8832u^4 + 226000u^3 - 90336u^2 - 1045232u + 1152429)/31928, \qquad 0 \leqq u \leqq 1,$$

$$M^2_{5,2,u} = \theta(u)(-20668u^6 + 155072u^5 - 311344u^4 - 113588u^3 + 679992u^2 - 154744u - 9523)/31928, \qquad 1 \leqq u \leqq 2,$$

$$M^2_{5,2,u} = \theta(u)(-31616u^6 + 474240u^5 - 2781708u^4 + 8057080u^3 - 12052136u^2 + 8947180u - 2637083)/31928, \qquad 2 \leqq u \leqq 3,$$

$$M^2_{5,2,u} = (5-u)u^2(-957840u^3 + 3152557u^2 - 3350090u + 1152429)/19156800, \qquad u \leqq 0.$$

In the above formulas the end coefficients a_0, a_m do not vanish in their range of validity except when the displayed linear factors vanish.

4. Derivations. Consider an approximation in $\mathfrak{A}_{m,n,u}$. For convenience we take $m \geqq 1$, leaving the case $m = 0$ to be treated by the reader. The relations (3) imply that

$$(8)\quad \begin{cases} k(t', u) = -\varphi_{t'}(u), & \text{if } t' < 0, \\ k(t', u) = \psi_{t'}(u) - (1-t')^n a_1/n! - (2-t')^n a_2/n! & \\ \qquad - \cdots - (m-t')^n a_m/n!, & \text{if } 0 \leqq t' < 1, \\ k(t', u) = \psi_{t'}(u) - (2-t')^n a_2/n! - \cdots - (m-t')^n a_m/n!, & \text{if } 1 \leqq t' < 2, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots & \\ k(t', u) = \psi_{t'}(u) - (m-t')^n a_m/n!, & \text{if } m-1 \leqq t' < m, \\ k(t', u) = \psi_{t'}(u), & \text{if } m \leqq t'. \end{cases}$$

Accordingly the integral J defined in (5) is the following function of $a_1, \cdots, a_m$:

$$J = \alpha - 2 \sum_{i=1,\cdots,m} \beta_i a_i + \sum_{i,j=1,\cdots,m} \gamma_{i,j} a_i a_j, \tag{9}$$

where

$$\gamma_{i,j} = \gamma_{j,i} = \int_0^i (i-t)^n (j-t)^n \, dt/(n!)^2 = \int_0^i s^n (s+j-i)^n \, ds/(n!)^2 \quad \text{if } i \leqq j;$$

$$\beta_i = \int_0^i \psi_i(u)(i-t)^n \, dt/n! = \begin{cases} 0 & \text{if } u \leqq 0, \\ \displaystyle\int_0^u (i-t)^n (u-t)^n \, dt/(n!)^2 & \\ \quad = \displaystyle\int_0^u s^n (s+i-u)^n \, ds/(n!)^2 & \text{if } 0 \leqq u \leqq i, \\ \displaystyle\int_0^i (i-t)^n (u-t)^n \, dt/(n!)^2 & \\ \quad = \displaystyle\int_0^i s^n (s+u-i)^n \, ds/(n!)^2 & \text{if } i \leqq u; \end{cases}$$

$$\alpha = \int_{\min(u,0)}^{\max(u,0)} (u-t)^{2n} \, dt/(n!)^2 = |u|^{2n+1}/[n!^2(2n+1)].$$

Lemma. The quantities $\alpha, \beta_i, \gamma_{i,j}$ are independent of m.

The quantities $\gamma_{i,j}$ are independent of u.

Define $\lambda(y, z)$ as $(-1)^n/(2n+1)!$ times the sum of the first $n+1$ terms of the binomial expansion of $(y-z)^{2n+1}$. Then $\gamma_{i,j} = \lambda(i, j)$ for $i \leqq j$; $\beta_i = 0$ for $u \leqq 0$, $\beta_i = \lambda(u, i)$ for $0 \leqq u \leqq i$, $\beta_i = \lambda(i, u)$ for $i \leqq u$.

As functions of u, α consists of two polynomial arcs and is of class C^{2n}; β_i consists of three polynomial arcs and is of class C^n at $u = 0$, of class C^{2n} at $u = i$.

The symmetric matrix $\| \gamma_{i,j} \|$ is positive definite.

For example, if $n = 1$, $6\gamma_{i,j} = 6\gamma_{j,i} = 3i^2 j - i^3$, $i \leqq j$; $6\beta_i = -u^3 + 3u^2 i$ for $u \leqq i$, $6\beta_i = 3i^2 u - i^3$ for $i \leqq u$.

That the matrix $\| \gamma_{i,j} \|$ is positive definite may be seen as follows: The quadratic term in (9) is what $\int_0^m k(t, u)^2 \, dt$ would equal if $\psi_i(u)$ in (8) were replaced by zero.

In order for the approximation A to be exact for polynomials of degree n, it is necessary and sufficient that it be exact for the $n+1$ polynomials: $x(t) = 1, t, t^2, \cdots, t^n$; that is, that the coefficients $a_0, \cdots, a_m$ be such that

$$a_0 + a_1 + \cdots + a_m = 1, \tag{10}$$

$$\sum_{i=1,\cdots,m} a_i i^\nu = u^\nu, \qquad \nu = 1, \cdots, n \quad \text{if } n \geqq 1. \tag{11}$$

We think of (10) as merely determining a_0 in terms of $a_1, \cdots, a_m$, since a_0 does not enter into (9).

Thus J is a quadratic function of $a_1, \cdots, a_m$ subject to the n constraints (11). The coefficients of $B_{m,n,u}$ are obtained by minimizing J subject to (11). One may show, by the use of Lagrange's multipliers, that the minimizing coefficients and J are given by the following relations. The actual calculation may also be carried through directly. We use a matrix notation:

$$a = \| a_i \|, \qquad \beta = \| \beta_i \|, \qquad v = \| u^\nu \|, \qquad i, j = 1, \cdots, m; \quad \nu = 1, \cdots, n,$$

$$H = \| i^\nu \|, \qquad \Gamma = \| \gamma_{i,j} \|,$$

where a, β, v each consist of one column, H consists of m rows and n columns, and Γ is square ($m \times m$). Since H is a Vandermonde matrix, its rank is maximal. The case $m < n$ is easily disposed of: Either the constraints (11) are inconsistent and $\mathfrak{A}_{m,n,u}$ is empty; or u is one of the integers $0, 1, \cdots, m$ and $\mathfrak{A}_{m,n,u}$ consists of the single element $A = x(u)$. Hereafter assume $m \geqq n$. Then H is of rank n. Since Γ is positive definite, the symmetric matrices

$$\Lambda = \Gamma^{-1}, \qquad E = (H^T \Lambda H)^{-1}$$

exist and are positive definite, where the superscript T indicates transposition. There is a unique minimizing a:

$$a = \Lambda\beta + \Lambda H E v - \Lambda H E H^T \Lambda\beta;$$

and the minimal J is

$$J_{\min} = \alpha - \beta^T \Lambda\beta + (v - H^T \Lambda\beta)^T E (v - H^T \Lambda\beta).$$

Hence, by the lemma, $J_{\min} = \alpha + v^T E v$ when $u \leqq 0$. Hence, also, the coefficients $a_0, \cdots, a_m$ in $B_{m,n,u}$ are broken polynomials (of degree $2n + 1$), with transitions at most at $u = 0, 1, \cdots, m$; and are of class C^{2n} on $0 \leqq u \leqq m$ and of class C^n for all u.

The square modulus of $B_{m,n,u}$ is

$$M^2_{m,n,u} = \begin{cases} (m - u) J_{\min} & \text{if } u < 0 \text{ and } a_m \neq 0, \\ m J_{\min} & \text{if } 0 \leqq u \leqq m \text{ and } a_0 a_m \neq 0, \\ u J_{\min} & \text{if } m < u \text{ and } a_0 \neq 0. \end{cases}$$

5. Linear transformation of the t-axis. Suppose that

$$t^* = bt + c, \qquad b \neq 0, \tag{12}$$

where b and c are constants; also that $x^*(t^*) = x(t)$; $u^* = bu + c$; $t_j^* = bj + c$ and $a_j^* = a_j^*(u^*) = a_j(u)$, $j = 0, 1, \cdots, m$; $R^*[x^*] = x^*(u^*) - \sum_{j=0,\cdots,m} a_j^* x^*(t_j^*)$; and that K^* is the (directed) image of K under (12). Then

$$R[x] = R^*[x^*] = \int_{K^*} x^{*(n+1)}(t^*) k^*(t^*, u^*)\, dt^*,$$

where $k^*(t^*, u^*) = b^n k(t, u)$; since $x^{*(n+1)}(t^*) = x^{(n+1)}(t)/b^{n+1}$. Hence

$$M^{*2} = |K^*| \int_{K^*} k^*(t^*, u^*)^2\, dt^* = |b|\,|K| \int_K k(t, u)^2\, dt\, b^{2n+1} = b^{2n+2} M^2.$$

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FORMULAS FOR NUMERICAL INTEGRATION OF FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS IN THE COMPLEX PLANE

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Although the literature on methods of stepwise numerical integration of differential equations in one or more real variables is quite voluminous, with detailed exposition of many formulas and processes, there has not yet appeared in print the simplest procedure for stepwise numerical integration of first or second order differential equations in the complex plane, save for the obvious and uneconomical method of expressing all functions and relations involving $z = x + iy$ as functions of x and y, and treating simultaneous systems of equations in x and y. Another theoretical disadvantage in expressing every differential equation in z as a simultaneous real system in x and y, is that we lose the advantage of a closer choice of points upon which to base the approximating polynomials. Thus suppose that a four-point formula is required when the integration proceeds stepwise in the real direction, as a function of x only. This means that the real or imaginary part of the integrand is considered as a polynomial which assumes values of that integrand at four points all along a line parallel to the x-axis, say at interval h, so that the first point will be $3h$ away from the last point where integration is performed. Now $3h$ is much greater than the furthest distance apart of four points at the corners of a square of side h, where any point of integration is less than $3h/2$ away from the other three points. For the nine-point configuration given below, the greatest distance between two points is less than three-eighths the greatest distance between two points had they been chosen equidistant along a straight line. Furthermore, there are many cases where n points along a straight line will not be adequate for an approximating polynomial for the integrand, whereas n points all taken nearer to each other will suffice. Finally, in calculations with analytic functions in the complex plane, it is generally more natural and expedient to retain z as the variable, rather than x and y. In particular, the essential property of analyticity that makes it suitable for computation, is that the functions behave as polynomials in z. Hence there follows the inevitably natural suggestion of deriving integration formulas based upon approximation of the function to be integrated as a polynomial in z—not the Taylor expansion polynomial, but the polynomial which assumes given functional values at given points, namely the Lagrangian interpolation polynomial. The latter usually possesses about the same accuracy as the Taylor expansion polynomial of the same degree.

The reason for tabulating formulas for obtaining a function from the values of its second derivative, as well as the single integration formulas, is the overwhelming importance of second-order differential equations, especially linear equations whose solution can be reduced to the solution of $d^2y/dz^2 + \phi(z)\, y = 0$. The formulas that are given below are for complex integration in a Cartesian grid, based upon three to nine points (i.e. assuming the integrand to behave as a polynomial

in z which may range from the second through the eighth degree in the vicinity of the fixed points). The choice of the configurations of the fixed points z_j was influenced by three considerations: I. Convenience of an L-shaped corner, which is also suggestive of initiation of the integration in either the x- or y-direction, beginning from a corner of the desired region in the z-plane. II. The points z_j should be as close to each other as possible. Thus in the five- and seven-point cases, the points z_j were not chosen to lie symmetrically about the 45° ray, even though that would have meant a lessening in the labor of deriving all the integration formulas, in order to obtain that slight theoretical advantage of closer points. III. The configuration should be such that as little mental effort as possible is required to picture it moved one space to the right, or one space upward, by extrapolations. The configurations of points that were decided upon for the various n-point cases are as follows:

Three-Point	Four-Point	Five-Point	Six-Point
$\dot{z}_i$ $\dot{z}_0\dot{z}_1$	$\dot{z}_i\dot{z}_{1+i}$ $\dot{z}_0\dot{z}_1$	$\dot{z}_i\dot{z}_{1+i}$ $\dot{z}_0\dot{z}_1\dot{z}_2$	$\dot{z}_{2i}$ $\dot{z}_i\dot{z}_{1+i}$ $\dot{z}_0\dot{z}_1\dot{z}_2$

Seven-Point	Eight-Point	Nine-Point
$\dot{z}_{2i}$ $\dot{z}_i\dot{z}_{1+i}\dot{z}_{2+i}$ $\dot{z}_0\dot{z}_1\dot{z}_2$	$\dot{z}_{2i}\dot{z}_{1+2i}$ $\dot{z}_i\dot{z}_{1+i}\dot{z}_{2+i}$ $\dot{z}_0\dot{z}_1\dot{z}_2$	$\dot{z}_{2i}\dot{z}_{1+2i}\dot{z}_{2+2i}$ $\dot{z}_i\dot{z}_{1+i}\dot{z}_{2+i}$ $\dot{z}_0\dot{z}_1\dot{z}_2$

Here $z_j \equiv z_0 + jh$, where h is the length of the squares comprising the grid. If the integrand in the formulas is denoted by $f(z)$, the indefinite integral $\int f(z)dz$ is denoted by $F(z)$, and after two integrations, $\int F(z)dz$ is denoted by $G(z)$. In the integration formulas, f_j , F_j and G_j denote $f(z_j)$, $F(z_j)$ and $G(z_j)$ respectively.

The n-point formulas for F_j and G_j were obtained by integrating once and twice respectively, the n-point complex Lagrangian polynomial which equals f_k at z_k , the n points of the configuration, and then calculating those integrals for $z = z_j$. All n-point formulas are given exactly for polynomials of degree $n - 1$. In the single integration, the arbitrary constant A' is introduced (A' to distinguish it from the A occurring in the double integration), while in the double integration, the arbitrary linear term $Az_j + B$ is introduced. The constants are solved for in the usual manner. Thus for the single integration, the formula is applied at a point z_j where the single integral F_j is known, to determine A'. Similarly, to determine A and B in $Az_j + B$, apply the double quadrature formula at two different points $z_j = z_{j_1}$ and $z_j = z_{j_2}$ where F_{j_1} and F_{j_2} are known, and solve two linear equations for A and B.

In the formulas for F_j and G_j , if the coefficients of f_k are denoted by a_k^j and b_k^j respectively, the reader will observe that, in all cases, save for the five- and

seven-point formulas (the configurations of whose reference points z_k lack symmetry about the 45° ray), if $j \equiv j_1 + ij_2$, $k \equiv k_1 + ik_2$, then

$$a_{k_2+ik_1}^{j_2+ij_1} = i\overline{a_{k_1+ik_2}^{j_1+ij_2}}, \quad \text{and} \tag{1}$$

$$b_{k_2+ik_1}^{j_2+ij_1} = -\overline{b_{k_1+ik_2}^{j_1+ij_2}}.^{*} \tag{2}$$

Relations (1) and (2) were helpful in reducing the amount of tedious labor in calculating a_k^j and b_k^j.

To prove (1) and (2), we first show that if in the Lagrangian polynomial for $f_P \equiv f_{p+iq}$, the coefficient of $f_k \equiv f_{k_1+ik_2}$ is denoted by $d_k^P \equiv d_{k_1+ik_2}^{p+iq}$, so that in the Lagrangian polynomial for $f_{iP} \equiv f_{q+ip}$ the coefficient of $f_{ik} \equiv f_{k_2+ik_1}$ is denoted by $d_{ik}^{iP} \equiv d_{k_2+ik_1}^{q+ip}$, then

$$d_{k_2+ik_1}^{q+ip} = \overline{d_{k_1+ik_2}^{p+iq}}. \tag{3}$$

By definition,

$$d_{k_1+ik_2}^{p+iq} = \frac{\Pi'[(p+iq)-(p_n+iq_n)]}{\Pi'[(k_1+ik_2)-(p_n+iq_n)]},$$

where the product Π' is taken for all points $p_n + iq_n$ belonging to the configuration, except for $k_1 + ik_2$. In Π' consider, in both numerator and denominator, the pairs of factors associated with $p_n + iq_n$ and $q_n + ip_n$, whenever $p_n \neq q_n$ and $p_n + iq_n \neq k_2 + ik_1$. The existence of such pairs is by virtue of the symmetry of the configuration about the 45° ray. Obviously,

$$\frac{[(p+iq)-(p_n+iq_n)][(p+iq)-(q_n+ip_n)]}{[(k_1+ik_2)-(p_n+iq_n)][(k_1+ik_2)-(q_n+ip_n)]}$$

$$= \frac{[(q-ip)-(q_n-ip_n)][(q-ip)-(p_n-iq_n)]}{[(k_2-ik_1)-(q_n-ip_n)][(k_2-ik_1)-(p_n-iq_n)]}.$$

Since $\bar{Q}\,\bar{R}/\bar{S}\,\bar{T} = \overline{Q\,R/S\,T}$, the right-hand expression is

$$\text{(A)} \qquad \overline{\frac{[(q+ip)-(q_n+ip_n)][(q+ip)-(p_n+iq_n)]}{[(k_2+ik_1)-(q_n+ip_n)][(k_2+ik_1)-(p_n+iq_n)]}}.$$

For $p_n \neq q_n$ and $p_n + iq_n = k_2 + ik_1$, there is the single factor $\dfrac{[(p+iq)-(k_2+ik_1)]}{[(k_1+ik_2)-(k_2+ik_1)]}$ which does not occur in a pair. This is equal to

$$\text{(B)} \qquad \overline{\frac{[(q+ip)-(k_1+ik_2)]}{[(k_2+ik_1)-(k_1+ik_2)]}}.$$

Finally, whenever $p_n = q_n$, there is an unpaired factor in Π' of the form $\dfrac{[(p+iq)-(p_n+ip_n)]}{[(k_1+ik_2)-(p_n+ip_n)]}$ which is equal to

* More concisely, $a_{ik}^{i\bar{j}} = i\overline{a_k^j}$, $b_{ik}^{i\bar{j}} = -\overline{b_k^j}$. Generally, if c_k^j is the coefficient of f_k in the expression for the m^{th} integral at z_j, then $c_{ik}^{i^m\bar{j}} = (i)^m\overline{c_k^j}$.

(C) $$\frac{\overline{[(q+ip)-(p_n+ip_n)]}}{[(k_2+ik_1)-(p_n+ip_n)]}.$$

The product of all factors of the type (A), (B)[1], and (C), remembering that $\bar{Q}\,\bar{R}\,\bar{S}\cdots=\overline{Q\,R\,S\cdots}$, yields the conjugate of $d_{k_2+ik_1}^{q+ip}$, which is equivalent to (3). To prove (1), consider the coefficient of $f_{k_1+ik_2}$ in F_{p+iq}, which is $a_{k_1+ik_2}^{p+iq}$. If $d_{k_1+ik_2}^{p+iq}$, which is a function of the complex variable $P=p+iq$, is written as $d_{k_1+ik_2}(P)$ where P is now considered as a running variable of integration, then $a_{k_1+ik_2}^{p+iq}=\int_0^{p+iq} d_{k_1+ik_2}(P)\,dP$, and similarly $a_{k_2+ik_1}^{q+ip}=\int_0^{q+ip} d_{k_2+ik_1}(P)\,dP$. In the latter integral, changing the variable to Q by letting $P=i\bar{Q}$, one obtains $i\int_0^{p+iq} d_{k_2+ik_1}(i\bar{Q})d\bar{Q}$. This last integral, employing (3), is equal to

$$i\int_0^{p+iq}\overline{d_{k_1+ik_2}(Q)}\;\overline{dQ}\quad\text{or}\quad i\,\overline{\int_0^{p+iq} d_{k_1+ik_2}(Q)\,dQ}=i\,\overline{a_{k_1+ik_2}^{p+iq}},$$

which is equivalent to (1). Repetition of this preceding argument with $a_{k_1+ik_2}(P)$ in the integrand, instead of $d_{k_1+ik_2}(P)$, leads to (2).

The procedure for using the formulas given below for systematic integration in the complex plane, is quite obvious when one recalls the well-known similar processes for real variables. One commences with a few initial values of the integral together with the first or second derivatives, for a corner configuration of points z_j (these initial values are always obtainable from the differential equation and series expansions). By complex extrapolation, an approximate set of values for the first or second derivatives is found for a configuration of points z_j' which has the same shape as the initial configuration of points z_j, but all moved one interval h in either the x- or y-direction. The formulas for complex extrapolation which are given at the end, are for just certain of those new points z_j' which will suffice (together with some of the old points z_j) to repeat the initial configuration in either the x- or y-direction, so that the process can proceed systematically over any desired portion of the z-plane. The integration formulas are applied at all points of the new configuration, including one or two of those points which overlap with the earlier configuration, and where the single or double integral is known. This overlapping is necessary for the determination of the constants A', or A and B. To check the correctness or adequacy of the answers, the process is iterated, using newer and more accurate first or second derivatives that can be had from the newly obtained single or double integral and from the differential equation. If the second set of new values of the integral, obtained by the quadrature formulas and by solving for the constants from those overlapping values that were used previously, should agree with the first new set, their correctness may be assumed and no further iteration is necessary. Then the process of extrapolation for the derivatives, quadrature, solving for constants, (including iteration of the entire integration process until two successive sets of answers agree) is performed on the next configuration of points z_j'', and so forth,

[1] A factor of the type (B) does not occur when $k_1=k_2$.

until the end of the row is reached, and we are ready to begin the next row in a similar fashion.

For double integration using the three-point formula, the user will have to slightly modify or complicate the process in order to solve for both A and B, since a translation of one step of the three-point configuration leads to only one overlapping point. One modification would be to employ both the L-shaped configuration and an inverted L-shaped configuration, namely Γ, both at the same time. Since this or something equivalent, will almost certainly involve more work than the use of the four-point formula for systematic double integration, one can say that the three-point formulas for double quadrature are generally useless (they are tabulated here for the sake of completeness, occupying only a few lines).

The user of these formulas should be reminded emphatically (to avoid error) that for each separate configuration of points z_j, the constants of integration A', or A and B, will always differ from corresponding constants in earlier configurations, so that on each occasion, they must be solved for as above described. The procedure for integration can also be varied in the following obvious way: Extrapolate for approximate new values of the integral in the new configuration of points z_j', and obtain the approximate values of the derivatives from the differential equation, instead of extrapolating directly for the derivatives. Incidentally, it is not necessary to proceed at steps of only one shift of the configuration of points z_j to the configuration of points z_j' whenever seven, eight, or nine points are involved. Speeding up the process to two shifts of the configuration in one step, is possible by "continued extrapolation" for the derivatives, namely extrapolation based upon some values that were themselves obtained by extrapolation. The accuracy in continued extrapolation, if Lagrangian, is identical with the accuracy obtained by extrapolating directly from the originally unextrapolated values.

The advantage in complex integration by this method lies in its absolute simplicity and the identical character of successive stages in the process, regardless of the special form of the differential equation. Thus these formulas are by nature especially suited to automatic computing machinery, whether operated by punch cards or electronically. It is conceivable that in a number of instances, a set of machine instructions can be drawn up so that the operator may have only to introduce the initial values, in order to end up with a completed table in the z-plane.

FORMULAS FOR COMPLEX INTEGRATION

Three-Poin

Single Integration

$$12\,F_0/h = 0 + A'$$
$$12\,F_1/h = (6 + 2i)f_0 + (5 - i)f_1 + (1 - i)f_i + A'$$
$$12\,F_i/h = (2 + 6i)f_0 + (-1 + i)f_1 + (-1 + 5i)f_i + A'$$

Double Integration

$$24\,G_0/h^2 = 0 + Az_0 + B$$
$$24\,G_1/h^2 = (8 + 2i)f_0 + (3 - i)f_1 + (1 - i)f_i + Az_1 + B$$
$$24\,G_i/h^2 = (-8 + 2i)f_0 + (-1 - i)f_1 + (-3 - i)f_i + Az_i + B$$

Four-Point

Single Integration

$$24\,F_0/h = 0 + A'$$
$$24\,F_1/h = (9+5i)f_0 + (9-5i)f_1 + (3+i)f_i + (3-i)f_{1+i} + A'$$
$$24\,F_i/h = (5+9i)f_0 + (1+3i)f_1 + (-5+9i)f_i + (-1+3i)f_{1+i} + A'$$
$$24\,F_{1+i}/h = (8+8i)f_0 + (4+4i)f_1 + (4+4i)f_i + (8+8i)f_{1+i} + A'$$

Double Integration

$$120\,G_0/h^2 = 0 + Az_0 + B$$
$$120\,G_1/h^2 = (33+13i)f_0 + (12-12i)f_1 + (8+2i)f_i + (7-3i)f_{1+i} + Az_1 + B$$
$$120\,G_i/h^2 = (-33+13i)f_0 + (-8+2i)f_1 + (-12-12i)f_i + (-7-3i)f_{1+i} + Az_i + B$$
$$120\,G_{1+i}/h^2 = 56if_0 + (4+20i)f_1 + (-4+20i)f_i + 24if_{1+i} + Az_{1+i} + B$$

Five-Point

Single Integration

$$600\,F_0/h = 0 + A'$$
$$600\,F_1/h = (195+95i)f_0 + (285-185i)f_1 + (6+18i)f_2 + (39+37i)f_i + (75+35i)f_{1+i} + A'$$
$$600\,F_2/h = (240+40i)f_0 + (720+80i)f_1 + (192-24i)f_2 + (48-16i)f_i - 80if_{1+i} + A'$$
$$600\,F_i/h = (155+195i)f_0 + (85+135i)f_1 + (-18+6i)f_2 + (-137+189i)f_i + (-85+75i)f_{1+i} + A'$$
$$600\,F_{1+i}/h = (200+140i)f_0 + (220+100i)f_1 + (-12+24i)f_2 + (52+76i)f_i + (140+260i)f_{1+i} + A'$$

Double Integration

$$1200\,G_0/h^2 = 0 + Az_0 + B$$
$$1200\,G_1/h^2 = (295+105i)f_0 + (170-190i)f_1 + (9+17i)f_2 + (46+38i)f_i + (80+30i)f_{1+i} + Az_1 + B$$
$$1200\,G_2/h^2 = (720+240i)f_0 + (1280-320i)f_1 + (112+16i)f_2 + (128+64i)f_i + 160f_{1+i} + Az_2 + B$$
$$1200\,G_i/h^2 = (-305+165i)f_0 + (-150+70i)f_1 + (-3-19i)f_2 + (-82-126i)f_i + (-60-90i)f_{1+i} + Az_i + B$$
$$1200\,G_{1+i}/h^2 = (60+500i)f_0 + (160+320i)f_1 + (-36+12i)f_2 + (-64+128i)f_i + (-120+240i)f_{1+i} + Az_{1+i} + B$$

Six-Point

Single Integration

$$480\,F_0/h = 0 + A'$$
$$480\,F_1/h = (139+95i)f_0 + (184-160i)f_1 + (8+7i)f_2 + (48+72i)f_i + (94-10i)f_{1+i} + (7-4i)f_{2i} + A'$$
$$480\,F_2/h = (208+80i)f_0 + (512+128i)f_1 + (144-32i)f_2 + 128f_i + (-32-160i)f_{1+i} - 16if_{2i} + A'$$
$$480\,F_i/h = (95+139i)f_0 + (72+48i)f_1 + (-4+7i)f_2 + (-160+184i)f_i + (-10+94i)f_{1+i} + (7+8i)f_{2i} + A'$$
$$480\,F_{1+i}/h = (124+124i)f_0 + (128+32i)f_1 + 12if_2 + (32+128i)f_i + (184+184i)f_{1+i} + 12f_{2i} + A'$$
$$480\,F_{2i}/h = (80+208i)f_0 + 128if_1 - 16f_2 + (128+512i)f_i + (-160-32i)f_{1+i} + (-32+144i)f_{2i} + A'$$

Double Integration

$$3360\,G_0/h^2 = 0 + Az_0 + B$$
$$3360\,G_1/h^2 = (773+375i)f_0 + (304-552i)f_1 + (33+18i)f_2 + (216+256i)f_i + (330-78i)f_{1+i} + (24-19i)f_{2i} + Az_1 + B$$

$$3360\,G_2/h^2 = (1952 + 960i)f_0 + (3072 - 768i)f_1 + (304 - 48i)f_2 + (768 + 512i)f_i + (576 - 576i)f_{1+i} + (48 - 80i)f_{2i} + Az_2 + B$$

$$3360\,G_i/h^2 = (-773 + 375i)f_0 + (-216 + 256i)f_1 + (-24 - 19i)f_2 + (-304 - 552i)f_i + (-330 - 78i)f_{1+i} + (-33 + 18i)f_{2i} + Az_i + B$$

$$3360\,G_{1+i}/h^2 = 1296if_0 + (480 + 544i)f_1 + (-40 + 48i)f_2 + (-480 + 544i)f_i + 880if_{1+i} + (40 + 48i)f_{2i} + Az_{1+i} + B$$

$$3360\,G_{2i}/h^2 = (-1952 + 960i)f_0 + (-768 + 512i)f_1 + (-48 - 80i)f_2 + (-3072 - 768i)f_i + (-576 - 576i)f_{1+i} + (-304 - 48i)f_{2i} + Az_{2i} + B$$

Seven-Point

Single Integration

$$3360\,F_0/h = 0 + A'$$

$$3360\,F_1/h = (835 + 589i)f_0 + (1164 - 1548i)f_1 + (123 - 40i)f_2 + (24 + 548i)f_i + (1058 + 510i)f_{1+i} + (116 - 80i)f_{2+i} + (40 + 21i)f_{2i} + A'$$

$$3360\,F_2/h = (1040 + 688i)\,f_0 + (2496 + 320i)f_1 + (960 - 528i)f_2 + (384 + 704i)f_i + (1696 - 800i)f_{1+i} + (64 - 384i)f_{2+i} + 80f_{2i} + A'$$

$$3360\,F_i/h = (687 + 817i)f_0 + (860 + 68i)f_1 + (79 + 80i)f_2 + (-1272 + 1012i)f_i + (-470 + 1238i)f_{1+i} + (116 + 80i)f_{2+i} + 65if_{2i} + A'$$

$$3360\,F_{1+i}/h = (780 + 692i)f_0 + (1072 - 304i)f_1 + (132 + 40i)f_2 + (-128 + 720i)f_i + (1288 + 2168i)f_{1+i} + 176f_{2+i} + (40 + 44i)f_{2i} + A'$$

$$3360\,F_{2+i}/h = (831 + 833i)f_0 + (1116 + 388i)f_1 + (143 + 224i)f_2 + (8 + 1044i)f_i + (3466 + 438i)f_{1+i} + (1076 + 368i)f_{2+i} + (80 + 65i)f_{2i} + A'$$

$$3360\,F_{2i}/h = (912 + 1200i)f_0 + (1216 + 1088i)f_1 + (32 + 272i)f_2 + (1152 + 2752i)f_i + (-3040 + 96i)f_{1+i} + (64 + 384i)f_{2+i} + (-336 + 928i)f_{2i} + A'$$

Double Integration

$$6720\,G_0/h^2 = 0 + Az_0 + B$$

$$6720\,G_1/h^2 = (1383 + 699i)f_0 + (384 - 1532i)f_1 + (118 - 73i)f_2 + (110 + 626i)f_i + (1210 + 374i)f_{1+i} + (106 - 110i)f_{2+i} + (49 + 16i)f_{2i} + Az_1 + B$$

$$6720\,G_2/h^2 = (3264 + 1920i)f_0 + (4864 - 2816i)f_1 + (672 - 544i)f_2 + (512 + 1792i)f_i + (3712 + 128i)f_{1+i} + (256 - 512i)f_{2+i} + (160 + 32i)f_{2i} + Az_2 + B$$

$$6720\,G_i/h^2 = (-1385 + 807i)f_0 + (-224 + 948i)f_1 + (-104 + 69i)f_2 + (-282 - 1206i)f_i + (-1190 - 706i)f_{1+i} + (-110 + 106i)f_{2+i} + (-65 - 18i)f_{2i} + Az_i + B$$

$$6720\,G_{1+i}/h^2 = (96 + 2304i)f_0 + (1728 + 704i)f_1 + (112 + 192i)f_2 + (-1152 + 512i)f_i + (-960 + 2720i)f_{1+i} + (192 + 192i)f_{2+i} + (-16 + 96i)f_{2i} + Az_{1+i} + B$$

$$6720\,G_{2+i}/h^2 = (1683 + 3819i)\,f_0 + (3856 + 708i)f_1 + (372 + 421i)f_2 + (-1338 + 2266i)f_i + (4482 + 5526i)f_{1+i} + (930 + 490i)f_{2+i} + (95 + 210i)f_{2i} + Az_{2+i} + B$$

$$6720\,G_{2i}/h^2 = (-3392 + 2304i)f_0 + (-1280 + 2816i)f_1 + (-416 + 160i)f_2 + (-4864 - 1536i)f_i + (-2432 - 3712i)f_{1+i} + (-512 + 256i)f_{2+i} + (-544 - 288i)f_{2i} + Az_{2i} + B$$

Eight-Point

Single Integration

$$6720\,F_0/h = 0 + A'$$

$$6720\,F_1/h = (1421 + 1197i)f_0 + (1562 - 3288i)f_1 + (112 - 195i)f_2 + (-680 + 1402i)f_i + (4606 + 830i)f_{1+i} + (-36 - 390i)f_{2+i} + (-35 + 176i)f_{2i} + (-230 + 268i)f_{1+2i} + A'$$

$$6720\,F_2/h = (1568 + 1568i)f_0 + (3264 + 704i)f_1 + (1568 - 1216i)f_2 + (-576 + 2496i)f_i + (8512 - 3520i)f_{1+i} + (-576 - 1088i)f_{2+i} + 352if_{2i} + (-320 + 704i)f_{1+2i} + A'$$

$$6720\,F_i/h = (1197 + 1421i)f_0 + (1402 - 680i)f_1 + (176 - 35i)f_2 + (-3288 + 1562i)f_i + (830 + 4606i)f_{1+i} + (268 - 230i)f_{2+i} + (-195 + 112i)f_{2i} + (-390 - 36i)f_{1+2i} + A'$$

$$6720\,F_{1+i}/h = (1288 + 1288i)f_0 + (1424 - 1168i)f_1 + (176 - 104i)f_2 + (-1168 + 1424i)f_i + (5296 + 5296i)f_{1+i} + (176 - 368i)f_{2+i} + (-104 + 176i)f_{2i} + (-368 + 176i)f_{1+2i} + A'$$

$$6720\,F_{2+i}/h = (1197 + 1421i)f_0 + (1082 - 424i)f_1 + (176 + 93i)f_2 + (-1624 + 1818i)f_i + (11582 + 3326i)f_{1+i} + (1932 + 26i)f_{2+i} + (-195 + 240i)f_{2i} + (-710 + 220i)f_{1+2i} + A'$$

$$6720\,F_{2i}/h = (1568 + 1568i)f_0 + (2496 - 576i)f_1 + 352f_2 + (704 + 3264i)f_i + (-3520 + 8512i)f_{1+i} + (704 - 320i)f_{2+i} + (-1216 + 1568i)f_{2i} + (-1088 - 576i)f_{1+2i} + A'$$

$$6720\,F_{1+2i}/h = (1421 + 1197i)f_0 + (1818 - 1624i)f_1 + (240 - 195i)f_2 + (-424 + 1082i)f_i + (3326 + 11582i)f_{1+i} + (220 - 710i)f_{2+i} + (93 + 176i)f_{2i} + (26 + 1932i)f_{1+2i} + A'$$

Double Integration

$$100800\,G_0/h^2 = 0 + Az_0 + B$$

$$100800\,G_1/h^2 = (18691 + 11101i)f_0 + (-1018 - 23186i)f_1 + (435 - 1814i)f_2 + (-3896 + 13292i)f_i + (38690 - 550i)f_{1+i} + (-1080 - 3088i)f_{2+i} + (16 + 1575i)f_{2i} + (-1438 + 2670i)f_{1+2i} + Az_1 + B$$

$$100800\,G_2/h^2 = (41792 + 31232i)f_0 + (49024 - 42112i)f_1 + (5280 - 10528i)f_2 + (-11392 + 41344i)f_i + (127360 - 22400i)f_{1+i} + (-5760 - 12416i)f_{2+i} + (32 + 5280i)f_{2i} + (-4736 + 9600i)f_{1+2i} + Az_2 + B$$

$$100800\,G_i/h^2 = (-18691 + 11101i)f_0 + (3896 + 13292i)f_1 + (-16 + 1575i)f_2 + (1018 - 23186i)f_i + (-38690 - 550i)f_{1+i} + (1438 + 2670i)f_{2+i} + (-435 - 1814i)f_{2i} + (1080 - 3088i)f_{1+2i} + Az_i + B$$

$$100800\,G_{1+i}/h^2 = 31232if_0 + (24928 - 864i)f_1 + (2624 + 496i)f_2 + (-24928 - 864i)f_i + 74080if_{1+i} + (4768 - 1888i)f_{2+i} + (-2624 + 496i)f_{2i} + (-4768 - 1888i)f_{1+2i} + Az_{1+i} + B$$

$$100800\,G_{2+i}/h^2 = (18691 + 51363i)f_0 + (44100 - 13700i)f_1 + (5264 + 77i)f_2 + (-45654 + 22778i)f_i + (132770 + 142110i)f_{1+i} + (13318 - 5126i)f_{2+i} + (-4813 + 3466i)f_{2i} + (-12476 + 632i)f_{1+2i} + Az_{2+i} + B$$

$$100800\,G_{2i}/h^2 = (-41792 + 31232i)f_0 + (11392 + 41344i)f_1 + (-32 + 5280i)f_2 + (-49024 - 42112i)f_i + (-127360 - 22400i)f_{1+i} + (4736 + 9600i)f_{2+i} + (-5280 - 10528i)f_{2i} + (5760 - 12416i)f_{1+2i} + Az_{2i} + B$$

$$100800\,G_{1+2i}/h^2 = (-18691 + 51363i)f_0 + (45654 + 22778i)f_1 + (4813 + 3466i)f_2 + (-44100 - 13700i)f_i + (-132770 + 142110i)f_{1+i} + (12476 + 632i)f_{2+i} + (-5264 + 77i)f_{2i} + (-13318 - 5126i)f_{1+2i} + Az_{1+2i} + B$$

Nine-Point

Single Integration

$$201600\,F_0/h = 0 + A'$$

$$201600\,F_1/h = (37891 + 33795i)f_0 + (27904 - 107100i)f_1 + (-1379 - 7965i)f_2 + (-39356 + 33600i)f_i + (232960 + 67200i)f_{1+i} + (-20036 - 20160i)f_{2+i} + (-5789 + 3165i)f_{2i} + (-25856 - 420i)f_{1+2i} + (-4739 - 2115i)f_{2+2i} + A'$$

$$201600\,F_2/h = (36512 + 41760i)f_0 + 55808f_1 + (36512 - 41760i)f_2 + (-59392 + 53760i)f_i + 465920f_{1+i} + (-59392 - 53760i)f_{2+i} + (-10528 + 5280i)f_{2i} - 51712f_{1+2i} + (-10528 - 5280i)f_{2+2i} + A'$$

$$201600\,F_i/h = (33795 + 37891i)f_0 + (33600 - 39356i)f_1 + (3165 - 5789i)f_2 + (-107100 + 27904i)f_i + (67200 + 232960i)f_{1+i} + (-420 - 25856i)f_{2+i} + (-7965 - 1379i)f_{2i} + (-20160 - 20036i)f_{1+2i} + (-2115 - 4739i)f_{2+2i} + A'$$

$$201600\,F_{1+i}/h = (34936 + 34936i)f_0 + (27904 - 49856i)f_1 + (1576 - 6824i)f_2 + (-49856 + 27904i)f_i + (232960 + 232960i)f_{1+i} + (-9536 - 25856i)f_{2+i} + (-6824 + 1576i)f_{2i} + (-25856 - 9536i)f_{1+2i} + (-3704 - 3704i)f_{2+2i} + A'$$

$$201600\,F_{2+i}/h = (33347 + 35971i)f_0 + (22208 - 39356i)f_1 + (2717 - 3869i)f_2 + (-58972 + 27904i)f_i + (398720 + 232960i)f_{1+i} + (47708 - 25856i)f_{2+i} + (-8413 + 541i)f_{2i} + (-31552 - 20036i)f_{1+2i} + (-2563 - 6659i)f_{2+2i} + A'$$

$$201600\,F_{2i}/h = (41760 + 36512i)f_0 + (53760 - 59392i)f_1 + (5280 - 10528i)f_2 + 55808if_i + 465920if_{1+i} - 51712if_{2+i} + (-41760 + 36512i)f_{2i} + (-53760 - 59392i)f_{1+2i} + (-5280 - 10528i)f_{2+2i} + A'$$

$$201600\,F_{1+2i}/h = (35971 + 33347i)f_0 + (27904 - 58972i)f_1 + (541 - 8413i)f_2 + (-39356 + 22208i)f_i + (232960 + 398720i)f_{1+i} + (-20036 - 31552i)f_{2+i} + (-3869 + 2717i)f_{2i} + (-25856 + 47708i)f_{1+2i} + (-6659 - 2563i)f_{2+2i} + A'$$

$$201600\,F_{2+2i}/h = (31232 + 31232i)f_0 + (2048 - 59392i)f_1 + (-5248 - 5248i)f_2 + (-59392 + 2048i)f_i + (465920 + 465920i)f_{1+i} + (-59392 + 2048i)f_{2+i} + (-5248 - 5248i)f_{2i} + (2048 - 59392i)f_{1+2i} + (31232 + 31232i)f_{2+2i} + A'$$

Double Integration

$$201600\,G_0/h^2 = 0 + Az_0 + B$$

$$201600\,G_1/h^2 = (34363 + 21699i)f_0 + (-14112 - 48384i)f_1 + (-2149 - 4131i)f_2 + (-19868 + 24572i)f_i + (137760 + 8960i)f_{1+i} + (-14236 - 8188i)f_{2+i} + (-2987 + 2647i)f_{2i} + (-14952 + 3328i)f_{1+2i} + (-3019 - 503i)f_{2+2i} + Az_1 + B$$

$$201600\,G_2/h^2 = (73024 + 59328i)f_0 + (55808 - 96768i)f_1 - 24192if_2 + (-65024 + 70144i)f_i + (465920 + 17920i)f_{1+i} + (-53760 - 37376i)f_{2+i} + (-10496 + 7424i)f_{2i} + (-51712 + 6656i)f_{1+2i} + (-10560 - 3136i)f_{2+2i} + Az_2 + B$$

$$201600\,G_i/h^2 = (-34363 + 21699i)f_0 + (19868 + 24572i)f_1 + (2987 + 2647i)f_2 + (14112 - 48384i)f_i + (-137760 + 8960i)f_{1+i} + (14952 + 3328i)f_{2+i} + (2149 - 4131i)f_{2i} + (14236 - 8188i)f_{1+2i} + (3019 - 503i)f_{2+2i} + Az_i + B$$

$$201600\,G_{1+i}/h^2 = 57776if_0 + (49856 - 20480i)f_1 + (5248 - 3696i)f_2 + (-49856 - 20480i)f_i + 241920if_{1+i} + (9536 - 22528i)f_{2+i} + (-5248 - 3696i)f_{2i} + (-9536 - 22528i)f_{1+2i} - 4688if_{2+2i} + Az_{1+i} + B$$

$$201600\,G_{2+i}/h^2 = (34251 + 93193i)f_0 + (75676 - 65532i)f_1 + (7397 - 9379i)f_2 + (-103832 + 7424i)f_i + (328160 + 474880i)f_{1+i} + (14112 - 48384i)f_{2+i} + (-12757 - 2601i)f_{2i} + (-37476 - 36868i)f_{1+2i} + (-3131 - 9533i)f_{2+2i} + Az_{2+i} + B$$

$$201600\,G_{2i}/h^2 = (-73024 + 59328i)f_0 + (65024 + 70144i)f_1 + (10496 + 7424i)f_2 + (-55808 - 96768i)f_i + (-465920 + 17920i)f_{1+i} + (51712 + 6656i)f_{2+i} - 24192if_{2i} + (53760 - 37376i)f_{1+2i} + (10560 - 3136i)f_{2+2i} + Az_{2i} + B$$

$$201600\,G_{1+2i}/h^2 = (-34251 + 93193i)f_0 + (103832 + 7424i)f_1 + (12757 - 2601i)f_2 + (-75676 - 65532i)f_i + (-328160 + 474880i)f_{1+i} + (37476 - 36868i)f_{2+i} + (-7397 - 9379i)f_{2i} + (-14112 - 48384i)f_{1+2i} + (3131 - 9533i)f_{2+2i} + Az_{1+2i} + B$$

$$201600\,G_{2+2i}/h^2 = 124928if_0 + (120832 - 55296i)f_1 + (10496 - 10496i)f_2 + (-120832 - 55296i)f_i + 931840if_{1+i} + (-2048 - 59392i)f_{2+i} + (-10496 - 10496i)f_{2i} + (2048 - 59392i)f_{1+2i} + Az_{2+2i} + B$$

FORMULAS FOR COMPLEX EXTRAPOLATION

Three-Point

$$f_2 = (-1 - 2i)f_0 + (3 + i)f_1 + (-1 + i)f_i$$

$$f_{1+i} = f_0 + if_1 - if_i$$

$$f_{2i} = (-1 + 2i)f_0 + (-1 - i)f_1 + (3 - i)f_i$$

Four-Point

$$f_2 = (2 - i)f_0 + (2 + 4i)f_1 - 2if_i + (-3 - i)f_{1+i}$$

$$f_{2+i} = 2if_0 + (-3 + i)f_1 + (2 + i)f_i + (2 - 4i)f_{1+i}$$

$$f_{1+3i} = -2if_0 + (2-i)f_1 + (-3-i)f_i + (2+4i)f_{1+i}$$
$$f_{2i} = (2+i)f_0 + 2if_1 + (2-4i)f_i + (-3+i)f_{1+i}$$

Five-Point

$$f_3 = -5f_0 - 15if_1 + (6+3i)f_2 + (-3+3i)f_i + (3+9i)f_{1+i}$$
$$f_{3+i} = f_0 + (1+3i)f_1 + (-\tfrac{4}{5}+\tfrac{2}{5}i)f_2 + (\tfrac{4}{5}-\tfrac{3}{5}i)f_i + (-1-3i)f_{1+i}$$
$$f_{1+2i} = (-2-i)f_0 - 5if_1 + f_2 + (-3+i)f_i + (5+5i)f_{1+i}$$
$$f_{2i} = (3-i)f_0 + (4+4i)f_1 + (-\tfrac{4}{5}+\tfrac{2}{5}i)f_2 + (\tfrac{4}{5}-\tfrac{26}{5}i)f_i + (-6+2i)f_{1+i}$$

Six-Point

$$f_3 = (-5-\tfrac{15}{2}i)f_0 + (12-21i)f_1 + (\tfrac{27}{4}+\tfrac{21}{4}i)f_2 + (-15-3i)f_i + (3+24i)f_{1+i} + (-\tfrac{3}{4}+\tfrac{9}{4}i)f_{2i}$$
$$f_{3+i} = (\tfrac{1}{2}+i)f_0 + (-1+3i)f_1 + (-\tfrac{3}{4}+\tfrac{1}{4}i)f_2 + (2+i)f_i - 5if_{1+i} + (\tfrac{1}{4}-\tfrac{1}{4}i)f_{2i}$$
$$f_{1+2i} = (\tfrac{1}{2}-i)f_0 + (2-i)f_1 + (\tfrac{1}{4}+\tfrac{1}{4}i)f_2 + (-1-3i)f_i + 5if_{1+i} + (-\tfrac{3}{4}-\tfrac{1}{4}i)f_{2i}$$
$$f_{2i} = (-5+\tfrac{15}{2}i)f_0 + (-15+3i)f_1 + (-\tfrac{3}{4}-\tfrac{9}{4}i)f_2 + (12+21i)f_i + (3-24i)f_{1+i} + (\tfrac{27}{4}-\tfrac{21}{4}i)f_{2i}$$

Seven-Point

$$f_3 = (\tfrac{11}{2}-\tfrac{3}{2}i)f_0 + (21+12i)f_1 + (\tfrac{3}{2}+12i)f_2 + (9-6i)f_i + (-27-21i)f_{1+i} + (-9+6i)f_{2+i} - \tfrac{3}{2}if_{2i}$$
$$f_{3+i} = (-\tfrac{5}{2}+\tfrac{3}{2}i)f_0 + (-18-6i)f_1 + (-\tfrac{9}{2}-6i)f_2 + (-8+6i)f_i + (30+15i)f_{1+i} + (6-12i)f_{2+i} + \tfrac{3}{2}if_{2i}$$
$$f_{3+2i} = (1+2i)f_0 + 8if_1 + (-1+2i)f_2 + (4+4i)f_i - 20if_{1+i} + (-4+4i)f_{2+i} + f_{2i}$$
$$f_{1+2i} = (-\tfrac{1}{2}-\tfrac{1}{2}i)f_0 + (-1-2i)f_1 - \tfrac{1}{2}if_2 + (-2-i)f_i + (5+5i)f_{1+i} - if_{2+i} - \tfrac{1}{2}f_{2i}$$
$$f_{3i} = (7+9i)f_0 + (6+30i)f_1 + (-3+6i)f_2 + (33+9i)f_i + (-42-54i)f_{1+i} + (-6+9i)f_{2+i} + (6-9i)f_{2i}$$

Eight-Point

$$f_3 = (4+6i)f_0 + (9+33i)f_1 + (-3+15i)f_2 + (12+18i)f_i + (-12-96i)f_{1+i} + (-18+12i)f_{2+i} + (3+3i)f_{2i} + (6+9i)f_{1+2i}$$
$$f_{3+i} = (-\tfrac{3}{2}-\tfrac{3}{2}i)f_0 + (-3-21i)f_1 - \tfrac{15}{2}if_2 + (-5-15i)f_i + 75if_{1+i} + (15-15i)f_{2+i} + (-\tfrac{3}{2}-3i)f_{2i} + (-3-9i)f_{1+2i}$$
$$f_{3+2i} = -f_0 - 4f_1 - f_2 - 4f_i + 20f_{1+i} - 4f_{2+i} - f_{2i} - 4f_{1+2i}$$
$$f_{1+2i} = (-\tfrac{3}{2}+\tfrac{3}{2}i)f_0 + (-5+15i)f_1 + (-\tfrac{3}{2}+3i)f_2 + (-3+21i)f_i - 75if_{1+i} + (-3+9i)f_{2+i} + \tfrac{15}{2}if_{2i} + (15+15i)f_{1+2i}$$
$$f_{2i} = (4-6i)f_0 + (12-18i)f_1 + (3-3i)f_2 + (9-33i)f_i + (-12+96i)f_{1+i} + (6-9i)f_{2+i} + (-3-15i)f_{2i} + (-18-12i)f_{1+2i}$$

Nine-Point

$$f_3 = (-\tfrac{7}{2}+\tfrac{9}{2}i)f_0 + (-21+27i)f_1 + (-\tfrac{21}{2}+\tfrac{27}{2}i)f_2 + (-18+12i)f_i + (138-66i)f_{1+i} + (-48+6i)f_{2+i} + (-\tfrac{9}{2}+\tfrac{3}{2}i)f_{2i} + (-24+3i)f_{1+2i} + (-\tfrac{15}{2}-\tfrac{3}{2}i)f_{3+2i}$$
$$f_{3+i} = (\tfrac{9}{4}-\tfrac{3}{4}i)f_0 + (12-6i)f_1 + (\tfrac{15}{4}-\tfrac{15}{4}i)f_2 + 10f_i - 75f_{1+i} + 30f_{2+i} + (\tfrac{3}{4}+\tfrac{3}{4}i)f_{2i} + (12+6i)f_{1+2i} + (\tfrac{15}{4}+\tfrac{15}{4}i)f_{2+2i}$$
$$f_{3+2i} = (-\tfrac{9}{2}-\tfrac{3}{2}i)f_0 + (-24-3i)f_1 + (-\tfrac{15}{2}+\tfrac{3}{2}i)f_2 + (-18-12i)f_i + (138+66i)f_{1+i} + (-48-6i)f_{2+i} + (-\tfrac{7}{2}-\tfrac{3}{2}i)f_{2i} + (-21-27i)f_{1+2i} + (-\tfrac{21}{2}-\tfrac{27}{2}i)f_{2+2i}$$
$$f_{2+2i} = (-\tfrac{9}{2}+\tfrac{3}{2}i)f_0 + (-18+12i)f_1 + (-\tfrac{7}{2}+\tfrac{9}{2}i)f_2 + (-24+3i)f_i + (138-66i)f_{1+i} + (-21+27i)f_{2+i} + (-\tfrac{15}{2}-\tfrac{3}{2}i)f_{2i} + (-48+6i)f_{1+2i} + (-\tfrac{21}{2}+\tfrac{27}{2}i)f_{2+2i}$$
$$f_{1+2i} = (\tfrac{9}{4}+\tfrac{3}{4}i)f_0 + 10f_1 + (\tfrac{9}{4}-\tfrac{3}{4}i)f_2 + (12+6i)f_i - 75f_{1+i} + (12-6i)f_{2+i} + (\tfrac{15}{4}+\tfrac{15}{4}i)f_{2i} + 30f_{1+2i} + (\tfrac{15}{4}-\tfrac{15}{4}i)f_{2+2i}$$
$$f_{2i} = (-\tfrac{7}{2}-\tfrac{9}{2}i)f_0 + (-18-12i)f_1 + (-\tfrac{9}{2}-\tfrac{3}{2}i)f_2 + (-21-27i)f_i + (138+66i)f_{1+i} + (-24-3i)f_{2+i} + (-\tfrac{21}{2}-\tfrac{27}{2}i)f_{2i} + (-48-6i)f_{1+2i} + (-\tfrac{15}{2}+\tfrac{3}{2}i)f_{2+2i}$$

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NUMERICAL QUADRATURE OF ANALYTIC AND HARMONIC FUNCTIONS

By Garrett Birkhoff and David Young

1. The Five Point Formula. We discuss below a new formula for performing numerical quadrature of real harmonic and complex analytic functions.[1]

Let $f(z)$ be an analytic function in the circle Ω: $|z - z_0| \leq h_1 > |h|$. Let the desired integral be (see Fig. 1):

$$I = \int_{z_1}^{z_1} f(z)\,dz \tag{1.1}$$

where $z_k = z_0 + (i)^{k-1}h$ $(k = 1, 2, 3, 4)$.[2]

Theorem 1: If $F(z)$ is any complex polynomial of the fifth degree or less:

$$I_1 = \int_{z_1}^{z_1} F(z)\,dz = \tfrac{1}{15}h[24F_0 + 4(F_1 + F_3) - (F_2 + F_4)] \tag{1.2}$$

where $F_i = F(z_i)$.

Proof: In the first place it is obvious that I_1 must be a linear function of the F_i. By symmetry we may write:

$$I_1 = aF_0 + b(F_1 + F_3) + c(F_2 + F_4). \tag{1.3}$$

If $F(z) = (z - z_0)^k$, formula (1.3) is exact for all odd k, for any values of a, b, c. It is exact for $k = 0, 2, 4$ if and only if:

$$a + 2b + 2c = 2h, \qquad b - c = h/3, \qquad b + c = h/5;$$

that is, if and only if $a = \frac{14}{15}h$, $b = \frac{4}{15}h$, $c = -\frac{1}{15}h$. By using the linearity of (1.3) we now get (1.2) for any complex polynomial of the form

$$F(z) = \sum_{k=0}^{5} a_k(z - z_0)^k$$

Hence, if we choose $F(z)$ as the polynomial approximation to $f(z)$ such that

$$F_k = f(z_k) = f_k \qquad\qquad k = 0, 1, 2, 3, 4$$

we have

$$\int_{z_1}^{z_1} f(z)\,dz \sim \int_{z_1}^{z_1} F(z)\,dz = \tfrac{1}{15}h[24f_0 + 4(f_1 + f_3) - (f_2 + f_4)]. \tag{1.4}$$

Formula (1.4) may be called the "five point formula" for complex quadrature.

[1] This formula was reported in a paper by Mr. Young, "On the Use of Conformal Mapping to Compute Flows with Free Streamlines"—Proceedings of the Symposium on Conformal Mapping, Institute of Numerical Analysis, held at Los Angeles in June, 1949. Other formulas are given in H. E. Salzer, "Formulas for Numerical Integration of First and Second Order Differential Equations in the Complex Plane", J. Math. Phys. **29**, 207. This work was done under Navy Contracts N5ori-76, Project 22 and N5ori-07634.

[2] In most practical cases h is chosen real or purely imaginary.

Since the coefficients of the f_i in (1.4) are real, the formula is equally applicable to the real part of f or in general to any harmonic function. This is discussed further in §4.

2. Error Estimate. By the use of Taylor's theorem for analytic functions, a first estimate of the error in (1.4) can be readily obtained.

Since $f(z)$ is analytic in $|z - z_0| \leq h_1 > |h|$ we may write the Taylor expansion of $f(z)$ absolutely and uniformly convergent in the closed region $|z - z_0| \leq |h|$:

$$f(z) = \sum_{\nu=1}^{\infty} c_\nu (z - z_0)^\nu, \quad \text{where } c_\nu = f^{(\nu)}(z_0)/\nu! \tag{2.1}$$

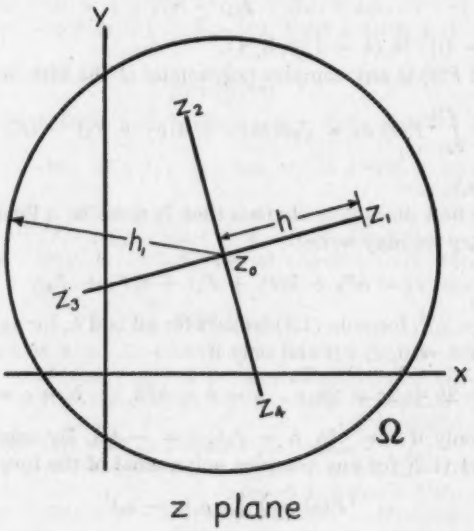

FIG. 1

Integrating term by term we get

$$I = \int_{z_1}^{z_1} f(z)\,dz = \sum_{\nu=0}^{\infty} c_\nu \int_{z_0-h}^{z_0+h} (z - z_0)^\nu\,dz$$

$$= \sum_{s=0}^{\infty} \frac{2h^{2s+1}}{2s+1} c_{2s}. \tag{2.2}$$

On the other hand, using (1.4) to integrate f we get

$$I_1 = \tfrac{1}{15}\{24c_0 + 4\sum_{\nu=1}^{\infty} c_\nu[h^\nu + (-h)^\nu] - \sum_{\nu=1}^{\infty} c_\nu[(ih)^\nu + (-ih)^\nu]$$

$$= \tfrac{1}{15}h\{24c_0 + \sum_{s=1}^{\infty} c_{2s}[8h^{2s} - 2(-1)^s h^{2s}]\}$$

$$I_1 = \tfrac{24}{15}hc_0 + \sum_{s=1}^{\infty} c_{2s} h^{2s+1} \tfrac{1}{15}[8 - 2(-1)^s] \tag{2.3}$$

since even powers of h vanish.

$$I_1 - I = \sum_{s=3}^{\infty} c_{2s} h^{2s+1} \left[\frac{8 - 2(-1)^s}{15} - \frac{2}{2s+1} \right] \tag{2.4}$$

$$= \tfrac{8}{315} c_6 h^7 + \tfrac{8}{945} c_8 h^9 + \tfrac{16}{3465} c_{10} h^{11} + \cdots$$

$$I_1 - I = \frac{f^{vi}(z_0)}{1890} h^7 + \frac{f^{viii}(z_0)}{8!} \tfrac{8}{945} h^9 + \cdots \tag{2.5}$$

$$= \frac{h^7 f^{vi}(z_0)}{1890} \left\{ 1 + \frac{1890 f^{viii}(z_0)}{8! f^{vi}(z_0)} \tfrac{8}{945} h^2 + \cdots \right\}$$

$$I_1 - I = \frac{h^7 f^{vi}(z_0)}{1890} [1 + G(z_0)] \tag{2.6}$$

where the series for $G(z_0)$ is obviously absolutely convergent.

Asymptotically for small h we have, therefore,

$$| I_1 - I | \sim \tfrac{1}{1890} | f^{vi}(z_0) | h |^7. \tag{2.7}$$

Finally, we note the following strict inequality:

$$| I_1 - I | \leq \frac{| h |^7}{1890} | f^{vi}(z) | \max \tag{2.7a}$$

where z takes on all values in the square with vertices z_1, z_2, z_3, z_4. The proof will be included in a later paper.

The error estimated by formula (2.7) is about four tenths as large as the corresponding error for the extended Simpson's rule:[3]

$$I_E = \int_{z_0-h}^{z_0+h} f(z)\, dz \sim \tfrac{1}{90} h \{ 114 f(z_0) + 34[f(z_0 + h) \tag{2.8}$$

$$+ f(z_0 - h)] - [f(z_0 + 2h) + f(z_0 - 2h)] \}$$

$$| I_E - I | \leq \tfrac{1}{756} | h^7 | | f^{vi}(\xi) |; \qquad \left(0 < \frac{\xi - (z_0 - 2h)}{4h} < 1 \right). \tag{2.9}$$

The five point formula is also better for integrating near singularities of $f(z)$ since the points are spaced nearer to z_0.

The five point formula is also seen to be more accurate, in theory, than Simpson's first rule:[4]

$$I_S = \int_{z_0-h}^{z_0+h} f(z)\, dz \sim \tfrac{1}{3} h [f(z_0 - h) + 4 f(z_0) + f(z_0 + h)] \tag{2.10}$$

$$| I_S - I | \sim \tfrac{1}{90} | h^5 | | f^{iv}(\xi) |; \qquad \left(0 < \frac{\xi - (z_0 - h)}{2h} < 1 \right).$$

3. Numerical Calculations. The estimates of the previous section regarding the relative accuracy of the five point formula, the extended Simpson's rule,

[3] Formula (2.8) is given in Scarborough, *Numerical Mathematical Analysis* (Johns Hopkins Press, 1930), p. 124. The error estimate (2.9) can readily be derived by using the methods of L. Ford, *Differential Equations*, p. 128.

[4] Scarborough, *loc. cit.*

and the ordinary Simpson's rule were verified numerically for some particular cases. The real parts of the functions $e^{1/z}$, $1/z$, $1/z^2$, $1/\log z$, $1/z \log z$ were integrated numerically from $z = .4$ to $z = .6$ using each formula and the results were compared with the exact values. It was found that the five point formula with $h = .1$ was approximately as accurate as Simpson's first rule with $h = .05$. The error in the extended Simpson's rule was about 2 or 3 times that of the five point formula, as expected.

The following general criteria may be given on the choice of integration rule to be used.

(i) The five point formula is preferable to Simpson's first rule when the integrand, $f(z)$, is difficult to evaluate, and is preferable to the extended Simpson's rule near singularities of $f(z)$.

(ii) When $f(z)$ is relatively easy to calculate the use of the simpler Simpson's first rule with a finer mesh may be indicated.

(iii) When $f(z)$ is fairly difficult to evaluate on a straight path Γ over which the integral is desired (for real values of z for example) but is much more difficult to evaluate for values of z not on Γ, the extended Simpson's rule may be preferable.

4. Numerical quadrature of harmonic functions. The five-point formula (1.4) can also be used advantageously to integrate real harmonic functions. Indeed, since the coefficients of (1.4) are real, and any harmonic function is the real part of a complex analytic function, the error estimates of §2 apply without change.

With this interpretation we obtain an interesting comparison with Simpson's Rule (2.10). By numerical comparison of coefficients, we have clearly

$$I_1 = I_S - \tfrac{1}{15}h\,R_0, \qquad \text{where} \tag{4.1}$$

$$R_0 = (f_1 + f_2 + f_3 + f_4 - 4f_0), \tag{4.2}$$

where f is a real harmonic or analytic complex function. In words, the five-point formula differs from Simpson's Rule by $h/15$ times the "residual" of $f(x, y)$, in the usual sense of the word.[5]

It follows that, if $f(x,y)$ is obtained by making the residual vanish identically, as is usual in numerical "solutions" of the Laplace equation by difference equation methods, Simpson's rule and our five-point formula give identical results!

To look at the matter another way, since $I_1 - I_S$ is by (2.7) the principal correction term for Simpson's Rule, for functions which are actually harmonic, the error committed in making five-point residuals (4.2) vanish is equivalent to neglecting this correction term. Hence, as long as the condition $R_0 = 0$ is used, there is presumably no gain in using any formula for numerical quadrature more elaborate than Simpson's rule (2.10).

[5] R. V. Southwell, *Relaxation Methods in Theoretical Physics*, (Oxford University Press, 1946).

It may be hoped that, if (4.1) is used to compute I, with functions which are actually harmonic, the smallness of R_0 will reliably indicate the accuracy of I_1. The argument then is, that since Simpson's Rule and the five-point formula agree well, both are accurate—and the five-point formula decidedly more accurate (cf. §3) than their difference.

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A STUDY OF THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS[1]

By George G. O'Brien, Morton A. Hyman and Sidney Kaplan

One of the most common and useful methods employed in the numerical integration of partial differential equations involves the replacement of the differential equation by an equivalent difference equation. This technique has become particularly important in recent years because of the development of modern high-speed computing machines.

In the present paper we shall show that the accuracy of a finite difference solution to a partial differential problem is conveniently discussed in terms of the "convergence" and "stability" of the difference scheme. Courant, Friedrichs, and Lewy (reference 1) discussed the convergence of difference solutions for the basic types of linear partial differential equations; for equations of parabolic or hyperbolic character, they found the important result that the "mesh-ratio" must satisfy certain inequalities. J. von Neumann obtained the same inequalities from a study of error-growth (stability of the difference scheme). The partly heuristic technique of stability analysis developed by von Neumann was applied by him to a wide variety of difference and differential equation problems during World War II. This method has been very briefly mentioned in the literature (references 2, 3, 4) but a detailed discussion has not yet been published. With the kind permission of Professor von Neumann, we have made such a discussion part of the present paper.

We begin with terminology and definitions. Let D represent the *exact solution* of the partial *differential* equation, Δ represent the *exact solution* of the partial *difference* equation, and N represent the *numerical solution* of the partial *difference* equation. We call $(D - \Delta)$ the truncation error; it arises because of the finite distance between points of the difference mesh. To find the conditions under which $\Delta \to D$ is the problem of *convergence*. We call $(\Delta - N)$ the numerical error. If a faultless computer working to an infinite number of decimal places were employed, the numerical error would be zero. Although $(\Delta - N)$ may consist of several kinds of errors, we usually consider it limited to round-off errors. To find the conditions under which $(\Delta - N)$ is small throughout the entire region of integration is the problem of *stability*.

Whether a given finite-difference scheme satisfies the criteria for convergence and stability (we say, for short, that the difference-scheme is convergent/divergent and stable/unstable) depends upon the form of the Δ-equation and upon the initial and boundary conditions. If the Δ-equation is linear, stability (and usually convergence also) will not depend on the initial and boundary conditions. Now for most problems, D and Δ are either unavailable or can only be obtained with much greater labor than is involved in finding N. The prin-

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cipal problem in the numerical solution of partial differential equations is to determine N such that $(D - N)$ is smaller than some preassigned allowable error throughout the whole region considered. We can assert that

$$(D - N) \equiv (D - \Delta) + (\Delta - N)$$

is small for a numerical calculation over a fine mesh using a stable, convergent difference scheme. Sometimes, for convenience or from necessity, a convergent but unstable difference scheme is used; then provision must be made for controlling the error-growth (see reference 9, where the numerical solution of elliptic problems is discussed; here the governing difference equations are inherently unstable).

In this paper we shall be interested, for the most part, in partial differential equations of parabolic or hyperbolic type, for which the data is naturally given on an open curve (or surface) from which the solution is stepped-off. Many remarks will apply, however, to elliptic problems. We shall mostly discuss equations of the second order in two independent variables, but extensions to equations of higher order and/or with more independent variables will be obvious, though algebraically more complex. In the first part of this paper, we shall give a method for determining the stability of partial difference equations and shall discuss "implicit" difference schemes. In the second part, we shall work with a simple parabolic problem and investigate directly the magnitudes of the truncation error and the numerical error for various methods of numerical solution; in particular, when $(D - N)$ is large, we shall ask whether *lack of convergence or lack of stability is chiefly responsible for the discrepancy*. We shall find that very often in such cases, the truncation error overshadows the numerical error, contrary to what is generally thought.

In studying the effect of round-off errors fed into the calculation (the problem of stability), we may ask:

(a) Does the over-all error due to all round-offs $\begin{pmatrix}\text{grow}\\ \text{not grow}\end{pmatrix}$? This we term *strong* $\begin{pmatrix}\text{instability}\\ \text{stability}\end{pmatrix}$.

(b) Does a single, general, round-off error $\begin{pmatrix}\text{grow}\\ \text{not grow}\end{pmatrix}$? This we term *weak* $\begin{pmatrix}\text{instability}\\ \text{stability}\end{pmatrix}$.

We mean "growth" during the *uninterrupted* stepping-ahead of the solution, where no use is made of special devices applied from time to time to limit the error growth (see end of second paragraph above). What we need to know in our numerical work is whether a given difference equation is strongly stable or strongly unstable. It is much easier, however, to demonstrate weak stability or instability. The gap between the two types of stability is closed by the following

Assumption: Weak $\begin{pmatrix}\text{stability}\\ \text{instability}\end{pmatrix}$ implies strong $\begin{pmatrix}\text{stability}\\ \text{instability}\end{pmatrix}$.

In the following text, then, whenever we refer to the stability or instability of a difference scheme we shall mean the weak form. We intend to examine rather closely in another paper the validity of this Assumption; for the present we note that it is true for all those calculations we have seen where care was taken that the round-off errors should be random.[2]

It is important to note that the over-all error may be considered as the sum of the individual errors fed in (modified from step to step by the numerical process) because the variational equation which governs error propagation is always linear and solutions may be superposed. For studying weak stability, we may adopt either of two procedures:

(i) Consider a unit error introduced at an arbitrary mesh point and follow its progress;

(ii) Make a Fourier expansion of a line of errors and follow the progress of the general term of the expansion.

The first procedure occurs occasionally in the literature but, to our knowledge, has not been developed in any systematic way; such a development has now been completed by R. P. Eddy (reference 6). The second procedure was developed and used by J. von Neumann during World War II, but has never been published by him. With his permission, we present below some of Professor von Neumann's results.

We begin with the simple heat equation

$$\partial\varphi/\partial t = \partial^2\varphi/\partial x^2, \tag{1}$$

which we replace with the difference equation

$$\frac{\varphi_{j,k+1} - \varphi_{j,k}}{\Delta t} = \frac{\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j-1,k}}{(\Delta x)^2}, \tag{2}$$

where

$$\varphi_{j,k} \equiv \varphi(x, t), \qquad \varphi_{j+1,k+1} \equiv \varphi(x + \Delta x, t + \Delta t), \qquad \text{etc.};$$

$$x = j\Delta x, \qquad t = k\Delta t.$$

In numerical work, we re-write (2) as

$$\varphi_{j,k+1} = \varphi_{j,k} + \Delta t/\Delta x^2\,[\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j-1,k}], \tag{2'}$$

which enables us to compute φ step-wise, line by line, starting from given initial values at $k = 0$ (conditions along certain boundaries must also be prescribed). If Δt is of the same order of magnitude as Δx^2, the error in the formula introduced by this replacement is of the order $(\Delta x)^2$ (see par. 21). Let the exact solution of the difference equation (Δ) be $\varphi(x, t)$ and call the solution actually

[2] As pointed out by Huskey and Hartree (reference 5), round-off errors may be non-random in certain regions of integration. They observe that randomness may be regained by carrying extra figures in calculating those regions. For general purposes, the assumption of random round-off errors is probably a good one.

obtained from numerical calculations (N) $\varphi(x, t) + \delta(x, t)$, where $\delta(x, t)$ is the error at the point (x, t) due to the accumulation of round-off errors. Since $\varphi(x, t) + \delta(x, t)$ must also satisfy (2):

$$\frac{(\varphi + \delta)_{j,k+1} - (\varphi + \delta)_{j,k}}{\Delta t} = \frac{(\varphi + \delta)_{j+1,k} - 2(\varphi + \delta)_{j,k} + (\varphi + \delta)_{j-1,k}}{(\Delta x)^2}. \tag{3}$$

Substracting (2) from (3), we find that δ must satisfy

$$\frac{\delta_{j,k+1} - \delta_{j,k}}{\Delta t} = \frac{\delta_{j+1,k} - 2\delta_{j,k} + \delta_{j-1,k}}{(\Delta x)^2}. \tag{4}$$

It will be observed that the variational equation (4) is the same as the difference equation (2) whose solution is Δ.[3] This is important in that it indicates that δ and the transient part of Δ have usually the same growth properties—i.e., δ and (trans Δ) increase or decrease together with t. We say "usually" rather than "invariably", because one must consider the effect of initial and boundary conditions, which influence (trans Δ) but not the error δ.

If we start with a nonlinear differential equation, or an equation with variable coefficients, we will still get a linear variational equation but its coefficients will depend on x, t, and (in the nonlinear case) Δ. The method of analysis which we give below (von Neumann's method) assumes constant coefficients. Following him, we circumvent the difficulty of non-constant coefficients by applying the method successively to a sequence of overlapping small regions—each region being so small that we may, with negligible error, assume the coefficients of the variational equation to be constant throughout the region; such a procedure, while it has worked well, is of course heuristic.

Consider the numerical solution of (2). At each line of computation a group of errors is introduced which propagates forward in accordance with (4). Recalling ii, of par. 7, we restrict attention to a single line of errors at $t = 0$. Call this line of errors $E(x)$ and make a harmonic decomposition of it:

$$E(x) = \sum_n A_n e^{i\beta_n x}. \tag{5}$$

This sum (5) must reduce to the correct error value at each of the mesh points on the line. The frequencies $|\beta_n|$ and the number of terms are somewhat arbitrary, but certain convenient choices can be made, which are the same for every line. For example, if the interval of interest in the x-direction has length $L = M\Delta x$ (M the number of subdivisions), it is convenient to have just as many terms as there are error-introducing mesh points (usually $M + 1$) with $|\beta| = 0, \omega, 2\omega, \cdots$; ω being the fundamental frequency, with period $2L$. Such a choice for the harmonic decomposition (5) allows the use of various elegant properties of finite trigonometric sums, in particular the orthogonality property:

$$\sum_{l=0}^{N-1} e^{i(lr\pi)/N} e^{-i(ls\pi)/N} = \begin{cases} N, & r = s \\ 0, & r \neq s \end{cases}$$

$$(r, s = 0, 1, 2, \cdots, N - 1)$$

[3] This will always happen when the original differential equation is linear.

with appropriate modifications if one uses the real harmonic form. If the boundary values are rigidly fixed so that they can introduce no error, the frequencies occurring can be taken as $l\pi/M\Delta x (l = 1, 2, \cdots, M-1)$. If there are no boundaries, one conveniently uses all frequencies between $|\beta| = 0$ and $|\beta| = \pi/\Delta x$; the finite sum (5) becomes an integral over this range of β. Because of superposition, we need only consider the single error term $e^{i\beta x}$, where β is any member of the sequence $\{\beta_n\}$. To allow for all possible subdivisions we must let β be any real number. We now seek a solution of (4) which reduces to $e^{i\beta x}$ when $t = 0$. There is one, and only one, such solution; it is

$$e^{\alpha t} e^{i\beta x},$$

where $\alpha = \alpha(\beta)$ is generally complex. Indeed, solutions of the form

$$\{e^{\alpha t} e^{i\beta x}\}$$

can be found for any linear homogeneous partial difference equation with constant coefficients and independent variables x, t; for an equation of the N'th order in t, there are N α's for each β. The functions $\{e^{\alpha t} e^{i\beta x}\}$ form a basic set of solutions for such an equation in the sense that any solution can be built up from members of the set (in general, it is necessary to add on at each line $t = k\Delta t$ a new linear combination of basic solutions). The power and simplicity of von Neumann's method for error analysis hinges on this fact. In order that the original error, $e^{i\beta x}$, shall not grow as t increases, it is necessary and sufficient that $|e^{\alpha\Delta t}| \leq 1$ (we repeat that β here is to have all real values, since in a general calculation all error frequencies can enter).

Let us now substitute $e^{\alpha t} e^{i\beta x}$ into (4). On setting $r = \Delta t/(\Delta x)^2$ and $\xi = e^{\alpha\Delta t}$ and simplifying slightly, we get

$$\begin{aligned} \xi - 1 &= r(e^{i\beta\Delta x} - 2 + e^{-i\beta\Delta x}) \\ &= 2r(\cos\beta\Delta x - 1) = -4r\sin^2(\beta\Delta x/2). \end{aligned}$$

The condition that $|\xi| \leqq 1$ yields

$$-1 \leqq 1 - 4r\sin^2(\beta\Delta x/2) \leqq 1.$$

The right inequality is satisfied trivially; the left inequality will be satisfied for all β if, and only if,

$$r \leqq \tfrac{1}{2}.$$

Thus $r = \frac{1}{2}$ separates the region of stability, where errors decay, from the region of instability, where some errors grow.[4]

[4] In some cases β does not take on all values and the limit of stability r_s is $> \frac{1}{2}$. Thus if there is only a finite number, M, of subdivisions in the x-direction, $\sin^2(\beta\Delta x/2) \leqq \sin^2[(M-1)\pi/2M]$, and the condition is

$$r \leq r_s = \tfrac{1}{2}\sin^2[(M-1)\pi/2M].$$

If $M = 10$, for example, $r \leq 0.513$, and as M is taken larger $r_s \to \frac{1}{2}$. To cover all cases, including those involving very small Δx and problems unbounded in x, we must use $r \leq \frac{1}{2}$. When speaking of the "largest stable value" of the mesh-ratio we shall mean that value of the mesh-ratio r which assures non-growth for *all* frequencies $|\beta|$.

We can also apply von Neumann's method to calculate the region of stability of hyperbolic partial differential equations. Using as our example

$$\partial^2\varphi/\partial t^2 = \partial^2\varphi/\partial x^2, \tag{6}$$

we take as the approximating difference equation:

$$\frac{\varphi_{j,k+1} - 2\varphi_{j,k} + \varphi_{j,k-1}}{(\Delta t)^2} = \frac{\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j-1,k}}{(\Delta x)^2}. \tag{7}$$

Replacing $\varphi_{j,k}$ by $e^{\alpha t}e^{i\beta x}$, setting $r \equiv \Delta t/\Delta x$, $\xi \equiv e^{\alpha\Delta t}$, we get:

$$\xi + 1/\xi = 2 - 4r^2 \sin^2(\beta\Delta x/2), \tag{8}$$

or

$$\xi^2 - 2A\xi + 1 = 0,$$

where

$$A \equiv 1 - 2r^2 \sin^2(\beta\Delta x/2).$$

The roots of the quadratic are

$$\xi_1 = A + \sqrt{A^2 - 1}, \qquad \xi_2 = A - \sqrt{A^2 - 1}, \tag{8'}$$

where from (8) we see that $\xi_1 = 1/\xi_2$. If $A > 1$, $|\xi_1| > 1$; if $A < -1$, $|\xi_2| > 1$; if $|A| \leqq 1$, $|\xi_1| = |\xi_2| = 1$. Thus, for stability, $-1 \leqq A \leqq 1$, or $-1 \leqq 1 - 2r^2 \sin^2(\beta\Delta x/2) \leqq 1$. The right side of the inequality is trivial; the left side is satisfied for all β if, and only if, $r \leqq 1$. This, then, is the criterion for stability. Courant, Friedrichs, and Lewy (reference 1) encountered this condition in their study of convergence and gave the following interpretation. The region of influence of a single point (say $x = 0$, $t = 0$) for the *difference* equation is bounded by the lines $t = \pm rx$. This triangular region must enclose the corresponding region of influence of the point for the *differential* equation, otherwise convergence of Δ to D as Δx, $\Delta t \to 0$ is clearly impossible (since some points would never be influenced that clearly must be influenced in the limit). Thus $r \leqq 1$ is a necessary condition for convergence; reference 1 proved that it is also sufficient.

There is one special case which we note in connection with the above analysis, i.e., where $\beta = 0$. In this case, $\xi_1 = \xi_2 = 1$ and the two solutions of (7), $e^{i\beta x}\xi_1$ and $e^{i\beta x}\xi_2$, are linearly dependent. In this case the two linearly *independent* solutions are 1 and t, and the general solution for $\beta = 0$ is $a + bt$, where a, b, are determined by the amplitudes of the $\beta = 0$ error waves on the first two lines of computation. If $b \neq 0$ we will have linear instability (as contrasted with the usual exponential instability).[5] Linear instability is not serious for two reasons:

[5] The frequency $\beta = 0$ enters only if the boundary points are in error (otherwise, $0 < |\beta| < \pi/\Delta x$; see par. 10). However, for a very small β, the behavior of the solution near $t = 0$ is closely the same as for $\beta = 0$. For small β, the amplitude of $e^{i\beta x}$ (a combination of two amplitudes) oscillates slowly and almost linearly (except near the extrema)

(a) The amplitude of the wave with frequency $\beta = 0$ on any line is the constant component of the error wave on that line; if we assume that the errors on any line are random, this amplitude is either zero or very small;
(b) An initial error (linearly unstable) grows the same bounded amount between $t = 0$ and $t = t_1$ (t_1 finite) regardless of Δt.

We have given a concise (but, we hope, lucid) account of von Neumann's technique of stability analysis. It is clear that the technique can be extended to linear equations with constant coefficients of higher order than the second, with more than two independent variables—and to systems of equations (in the case of systems, several new ideas enter; these, however, are easily discovered). For linear equations with variable coefficients, we may seek a basic set of solutions analogous to $\{e^{\alpha t}e^{i\beta x}\}$; however, it is generally simpler to follow the procedure suggested in par. 9 for non-linear equations and consider a succession of small sub-regions in each of which the linear variational equation has, to a good approximation, constant coefficients (we may calculate several sub-regions without changing our computing set-up, if we can find somehow and use the strictest stability criterion among those for the sub-regions traversed). Finally, we remark that such stability analysis can be used to test the convergence of iterative schemes (such as are used in solving elliptic problems) since successive iterations can be regarded as "time steps" and the error at any point as a transient which does or does not "damp out". For example, the simplest iteration scheme for solving the two-dimensional Laplace's equation has for its basic formula

$$u_{r,s}^{(n+1)} = \tfrac{1}{4}[u_{r-1,s}^{(n)} + u_{r,s-1}^{(n)} + u_{r,s+1}^{(n)} + u_{r+1,s}^{(n)}]$$

where n is the number of the iteration cycle. This formula is easily seen to be equivalent, for $r \equiv \Delta t/\Delta x^2 = \Delta t/\Delta y^2 = \frac{1}{4}$ to the "heat equation":

$$\frac{u_{r,s}^{n+1} - u_{r,s}^{n}}{\Delta t} = \frac{u_{r-1,s}^{n} - 2u_{r,s}^{n} + u_{r+1,s}^{n}}{\Delta x^2} + \frac{u_{r,s-1}^{n} - 2u_{r,s}^{n} + u_{r,s+1}^{n}}{\Delta y^2}$$

where $x = r\Delta x$, $y = s\Delta y$, $t = n\Delta t$. Many iteration schemes cannot, as here, be conveniently expressed in terms of a partial difference equation involving the "time"; still, whenever there is a fixed procedure, the trend of the errors from cycle to cycle can always be studied by the stability techniques we have described.

We have shown that the stability criteria for the difference equations (2) and (6) are $\Delta t/\Delta x^2 \leq \frac{1}{2}$ and $\Delta t/\Delta x \leq 1$, respectively. Having chosen a Δx suitable for our purposes, we are immediately restricted in our choice of Δt. It often happens that the Δt thus forced upon us is inconveniently small; we would like to find some way whereby we could use larger Δt's without sacrificing stability. A way of doing this in some cases is to "turn around" an unstable difference representation. Thus the difference equation (2), represented schematically as

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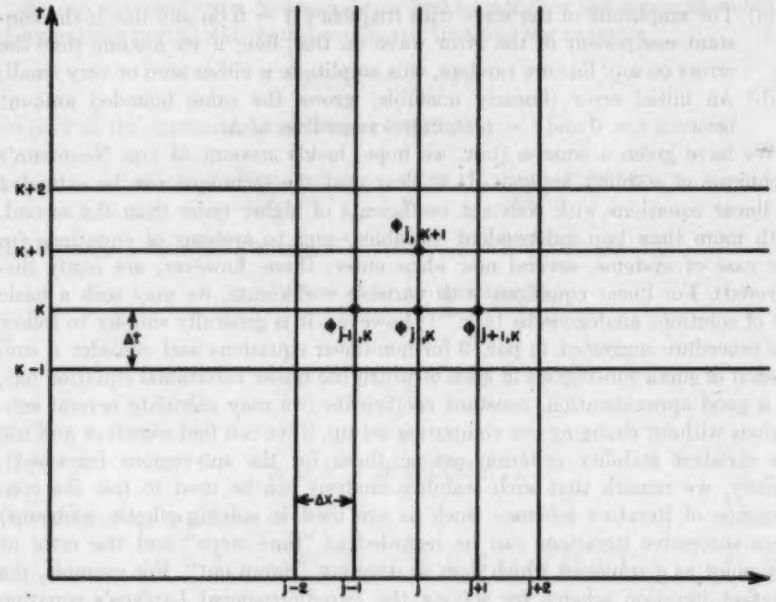
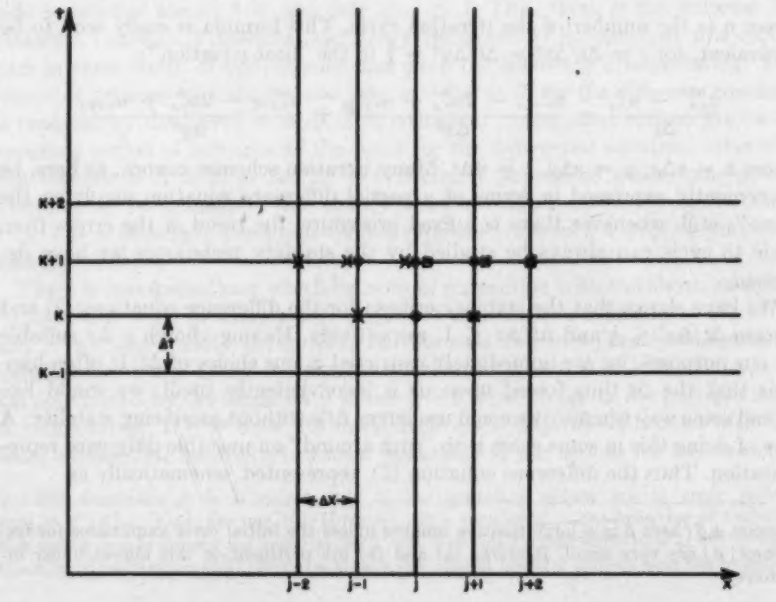
between $\pm S$; here S is a large positive number unless the initial error amplitudes for frequency $|\beta|$ are very small. Remarks (a) and (b) are pertinent to this almost-linear instability.



is stable only if $r \equiv \Delta t/(\Delta x)^2 \leq \frac{1}{2}$. However, if we turn the difference representation around,



we get decay of errors (i.e., stability) for all $r \geqq 0$, since the "growth factor" for frequency β is $\xi = 1/[1 + 4r \sin^2(\beta\Delta x/2)]$. It should be noted that the corresponding difference equation,

$$\frac{\varphi_{j,k+1} - \varphi_{j,k}}{\Delta t} = \frac{\varphi_{j+1,k+1} - 2\varphi_{j,k+1} + \varphi_{j-1,k+1}}{(\Delta x)^2},$$

is an implicit relation for the unknown functional values on line $(k + 1)$; that is, these unknowns occur simultaneously in the equations rather than singly as would be most convenient for computation.[6]

The use of implicit formulas is really the key to the problem of chosing Δt at the computer's convenience while retaining stability. D. R. Hartree and later English investigators (references 8 and 3) have used implicit formulas for solving parabolic problems. Von Neumann pointed out the stability advantages of such formulas, and has applied the implicit principle widely to both parabolic and hyperbolic problems. Let us follow him and replace the heat equation (1) by the difference equation

$$\frac{\varphi_{j,k+1} - \varphi_{j,k}}{\Delta t} = \frac{1}{2}\left\{\begin{array}{l} \dfrac{\varphi_{j+1,k+1} - 2\varphi_{j,k+1} + \varphi_{j-1,k+1}}{(\Delta x)^2} \\ + \dfrac{\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j+1,k}}{(\Delta x)^2} \end{array}\right\}. \tag{9}$$

As before, we replace $\varphi_{j,k}$ by $e^{i\beta x}e^{\alpha t}$ and set $\xi = e^{\alpha\Delta t}$, $r = \Delta t/(\Delta x)^2$. Then

$$\xi - 1 = \tfrac{1}{2}r[(\xi + 1)(-4 \sin^2(\beta\Delta x/2))]$$

or

$$\xi = 1 - 2r(\xi + 1) \sin^2(\beta\Delta x/2).$$

Solving for ξ, we obtain

$$\xi = \frac{1 - 2r \sin^2(\beta\Delta x/2)}{1 + 2r \sin^2(\beta\Delta x/2)}.$$

It is clear that, for any r and any β, $|\xi| \leqq 1$; hence (9) is always stable. Von Neumann replaces the wave equation (6) by the difference equation

$$\frac{\varphi_{j,k+1} - 2\varphi_{j,k} + \varphi_{j,k-1}}{(\Delta t)^2} = \left\{\begin{array}{l} a(\varphi_{j+1,k-1} - 2\varphi_{j,k-1} + \varphi_{j-1,k-1}) \\ + (1 - 2a)(\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j-1,k}) \\ + a(\varphi_{j+1,k+1} - 2\varphi_{j,k+1} + \varphi_{j-1,k+1}) \end{array}\right\} \Big/ (\Delta x)^2 \tag{10}$$

where a is any real number $\geqq 0$. As before, setting

$$\xi = e^{\alpha\Delta t}, \qquad r = \Delta t/\Delta x, \qquad A = r^2 \sin^2(\beta\Delta x/2),$$

[6] To amplify this remark, we consider the previous lattice. Each value on line k yields an equation in terms of the three points on line $(k + 1)$ nearest to it. The totality of equations, one for each point on line k, constitutes a simultaneous algebraic system, which we solve for the unknown values on line $(k + 1)$.

we easily obtain

$$\xi + 1/\xi - 2 = -4A[a(\xi + 1/\xi) + (1 - 2a)].$$

Solving for $(\xi + 1/\xi)$, we obtain $\xi + 1/\xi = 2B$, where $B = 1 - [2A/(1 + 4Aa)]$. We know from a previous calculation (par. 12) that $|\xi_1| \leqq 1$, $|\xi_2| \leqq 1$ if and only if $-1 \leqq B \leqq 1$. Thus $-1 \leqq 1 - 2A/(1 + 4Aa) \leqq 1$ or

$$1 \geqq A/(1 + 4Aa) \geqq 0.$$

Since A and a are $\geqq 0$, the right inequality is trivial. The left inequality, $1 + 4Aa \geqq A$, yields the two inequalities

$$A \leqq \frac{1}{1 - 4a}, \quad a \leqq \frac{1}{4}; \quad A > \frac{1}{1 - 4a}, \quad a > \frac{1}{4}.$$

The second inequality is trivial since $A = r^2 \sin^2(\beta\Delta x/2) \geqq 0$ always. Letting $\sin^2(\beta\Delta x/2)$ take on its largest possible value, we obtain from the first inequality

$$r = \Delta t/\Delta x \leqq 1/\sqrt{1 - 4a}, \qquad a < \tfrac{1}{4}.$$

For stability, the mesh-ratio $\Delta t/\Delta x$ must satisfy the above condition. If $a \geqq \frac{1}{4}$, one may choose $\Delta t/\Delta x$ as he pleases. Note that if $a = 0$, equation (10) reduces to equation (7) and $\Delta t/\Delta x \leqq 1$ as before. Von Neumann has suggested the possibility of changing a from time to time during the calculation for greater convenience in computing certain regions. Note that if $\beta = 0$, then $A = 0$, $B = 1$, and

$$\xi_1 = \xi_2 = 1.$$

Thus linear instability is possible for the implicit relation (10) as well as for the simple equation (7). Indeed, any difference representation for the wave equation (5) is subject to linear instability.

Consider the system of linear algebraic equations obtained by insisting that the condition (10) be satisfied at every point of the line k; the unknown quantities are the values $\varphi_{j,k+1}$ at the points of the line $(k + 1)$. This system of algebraic equations may be solved either directly or by successive approximations. In a direct solution, advantage may be taken of the simple and repetitive nature of the coefficient matrix; von Neumann has suggested that a "Green's function" be found—this would correspond to inverting the coefficient matrix once and for all.

There is, of course, more than one difference representation for a particular differential equation; however, not every such representation is stable. For example, instead of replacing the heat equation (1) by the difference representation (2) we could replace it by

$$\frac{\varphi_{j,k+1} - \varphi_{j,k-1}}{2\Delta t} = \frac{\varphi_{j+1,k} - 2\varphi_{j,k} + \varphi_{j-1,k}}{(\Delta x)^2}. \tag{11}$$

This form was used by L. F. Richardson in 1910 (his paper, reference 7, is now a classical reference in the numerical solution of partial differential equations).

We shall see (par. 21) that the error committed in the replacement of the t-derivative is very small, smaller than the error involved in the replacement (2). Nevertheless the difference equation (11) is completely unstable for all choices of Δx and $\Delta t(>0)$.

Replacing $\varphi_{j,k}$ by $e^{\alpha t}e^{i\beta x}$ in (11) we easily obtain

$$\xi - (1/\xi) = -8r \sin^2(\beta \Delta x/2), \tag{12}$$

where ξ and r have the same meanings as before. It is clear that the roots of this equation are negative reciprocals. The sum of the roots is $\xi + (-1/\xi)$. Unless $\beta = 0$, $\xi - 1/\xi < 0$ and either ξ or $-1/\xi$ is < -1. Since β must be assumed to be any real number, we conclude that Richardson's representation is unstable no matter how the mesh-ratio r is chosen.

Referring back to a remark made in par. 8, we see that the error growth just found for the difference equation (11) usually implies a similar growth of the transient portion of the solution of that equation. One might anticipate, therefore, that the solution of (11), for the given initial and boundary conditions, would oscillate with increasing amplitude about the solution of (1). That this expectation is borne out may be seen from Plate 1, which is a graphical comparison of the numerical solution of (11) with the exact solution of (1), the latter being

$$\varphi(x, t) = (4/\pi) \sum_{m=1,3,5,\cdots}^{\infty} m^{-1} e^{-m^2\pi^2 t} \sin(m\pi x). \tag{13}$$

The initial and boundary conditions imposed are $\varphi(x, 0) = 1$,

$$0 < x < 1; \qquad \varphi(0, t) = \varphi(1, t) = 0, \qquad t \geqq 0. \tag{13'}$$[7]

Physically, this problem represents the cooling of a finite bar with unit diffusivity —x being distance in centimeters and t time in seconds, say. Since Richardson carried his numerical work only as far as $t = 0.005$ seconds, Plate 1 shows how easy it was for him, without a knowledge of stability, to conclude incorrectly that (11) was a satisfactory difference scheme (see also Table I, columns 2 and 4).

We now compare Richardson's numerical results with those obtained from equation (2). Using the same mesh size as did Richardson, namely: $\Delta t = 0.001$, $\Delta x = 0.1$, we have the stable mesh-ratio $r = 0.1$. Equation (2) becomes

$$\varphi_{j,k+1} = 0.1(\varphi_{j+1,k} + \varphi_{j-1,k}) + 0.8\varphi_{j,k}. \tag{14}$$

Using this equation we computed Table II, column 4, and Table IV, columns 2 and 3. In using equation (2) we have purposely chosen a difference representation with a large error of approximation. Before comparing these numerical

[7] Richardson's solution is slightly different in form because he uses the accessory conditions $\varphi(x, 0) = 1$, $-0.5 < x < 0.5$; $\varphi(-0.5, t) = \varphi(+0.5, t) = 0$, $t \geqq 0$. Due to a typographical error the factor $4/\pi$ is missing from his expression.

results with Richardson's, let us compare the errors of approximation in the two difference equations. To do this apply Taylor's formula with a remainder:

$$\varphi(x - \Delta x, t) = \varphi(x, t) - \frac{\partial\varphi(x, t)\Delta x}{\partial x} + \frac{\partial^2\varphi(x, t)}{\partial x^2}\frac{(\Delta x)^2}{2!} - \frac{\partial^3\varphi(x, t)}{\partial x^3}\frac{(\Delta x)^3}{3!} + \frac{\partial^4\varphi(\xi_1, t)}{\partial x^4}\frac{(\Delta x)^4}{4!},$$

$$\varphi(x + \Delta x, t) = \varphi(x, t) + \frac{\partial\varphi(x, t)\Delta x}{\partial x} + \frac{\partial^2\varphi(x, t)}{\partial x^2}\frac{(\Delta x)^2}{2!} + \frac{\partial^3\varphi(x, t)}{\partial x^3}\frac{(\Delta x)^3}{3!} + \frac{\partial^4\varphi(\xi_2, t)}{\partial x^4}\frac{(\Delta x)^4}{4!},$$

$$\varphi(x, t - \Delta t) = \varphi(x, t) - \frac{\partial\varphi(x, t)\Delta t}{\partial t} + \frac{\partial^2\varphi(x, t)}{\partial t^2}\frac{(\Delta t)^2}{2!} - \frac{\partial^3\varphi(x, \xi_3)}{\partial t^3}\frac{(\Delta t)^3}{3!},$$

$$\varphi(x, t + \Delta t) = \varphi(x, t) + \frac{\partial\varphi(x, t)\Delta t}{\partial t} + \frac{\partial^2\varphi(x, t)}{\partial t^2}\frac{(\Delta t)^2}{2!} + \frac{\partial^3\varphi(x, \xi_4)}{\partial t^3}\frac{(\Delta t)^3}{3!},$$

or (15)

$$\varphi(x, t + \Delta t) = \varphi(x, t) + \frac{\partial\varphi(x, t)\Delta t}{\partial t} + \frac{\partial^2\varphi(x, \xi_3)}{\partial t^2}\frac{(\Delta t)^2}{2!},$$

where

$$x - \Delta x < \xi_1 < x < \xi_2 < x + \Delta x; \qquad t - \Delta t < \xi_3 < t < \xi_4, \quad \xi_3 < t + \Delta t.$$

The right sides of the equations in question (equations 11 and 2) are the same and, from (15)

$$\frac{\varphi(x + \Delta x, t) - 2\varphi(x, t) + \varphi(x - \Delta x, t)}{(\Delta x)^2} = \frac{\partial^2\varphi(x, t)}{\partial x^2} + \frac{(\Delta x)^2}{24}\left[\frac{\partial^4\varphi(\xi_1, t)}{\partial x^4} + \frac{\partial^4\varphi(\xi_2, t)}{\partial x^4}\right].$$

The error of approximation to the right side of $\partial\varphi/\partial t = \partial^2\varphi/\partial x^2$ is therefore of the order $(\Delta x)^2$. The left side of (11) is

$$\frac{\varphi(x, t + \Delta t) - \varphi(x, t - \Delta t)}{2\Delta t} = \frac{\partial\varphi(x, t)}{\partial t} + \frac{(\Delta t)^2}{12}\left[\frac{\partial^3\varphi(x, \xi_3)}{\partial t^3} + \frac{\partial^3\varphi(x, \xi_4)}{\partial t^3}\right].$$

The left side of (2) is

$$\frac{\varphi(x, t + \Delta t) - \varphi(x, t)}{\Delta t} = \frac{\partial\varphi(x, t)}{\partial t} + \frac{\Delta t}{2}\cdot\frac{\partial^2\varphi(x, \xi_3)}{\partial t^2}.$$

We see that the errors of approximation of the left sides of (11) and (2) are, respectively, of the order $(\Delta t)^2$ and Δt. Hence the solution of Richardson's

equation (11) might be *expected* to better represent D than the solution of (2). Due to stability considerations, we have seen that such is not the case. Table I gives D as well as Δ and N for the two difference equations. In this, and in all subsequent tables and graphs, $x = 0.4$ cm.

Little has been done toward comparing extensively numerical solutions of partial differential equations with the exact solutions. This is probably due, at least in part, to the tremendous numerical calculations involved. However, with modern computing machines this work is considerably lessened. We have studied in considerable detail the simple heat equation (1), making use of an I.B.M. tabulator #404 and desk computing machines to do the necessary long computations. It has permitted us to compare numerical solutions of the difference equation, using various mesh-ratios and mesh-sizes,

(a) with one another;

(b) with the exact solution (13) of the differential equation;

(c) with the exact solution of the difference equation for initial and boundary conditions (13′), namely:

$$\varphi(x, t) = \frac{2}{M} \sum_{m=1,3,5,\cdots}^{L} \cot \frac{\pi m}{2M} \sin (\pi m x) \left[1 - 4r \sin^2 \left(\frac{\pi m}{2M} \right) \right]^{M^2 t/r}; \quad (17)$$

here

$$M\Delta x = 1, \qquad L = \begin{cases} M - 1, M \text{ even} \\ M - 2, M \text{ odd} \end{cases}, \qquad x = j\Delta x, \qquad t = k\Delta t.$$

We turn aside briefly to remark that formula (17), like formula (18) following, is obtained by separating variables in the difference equation, then imposing the boundary conditions and the initial conditions (i.e., the value of the dependent variable at every interior mesh point of the line $k = 0$—or of the lines $k = 0, 1$). Both (17) and (18) can be put into convenient matrix form (see also reference 9); (17) is written

$$\varphi = CF$$

where

$$\varphi = \| \varphi_{jk} \|, \qquad C = \| C_{jm} \| = \left\| \sin \frac{\pi jm}{M} \right\|,$$

$$F = \| F_{mk} \|; \qquad F_{mk} = \begin{cases} \dfrac{2}{M} \cot \dfrac{\pi m}{2M} \left[1 - 4r \sin^2 \dfrac{\pi m}{2M} \right]^k, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$$

We note that $C^{-1} = \frac{2}{M} C$; i.e.,

$$\sum_{m=1}^{M-1} \sin \frac{\pi jm}{M} \sin \frac{\pi lm}{M} = \begin{cases} M/2, & j = l \\ 0, & j \neq l \end{cases}$$

$$(j, l = 1, 2, \cdots, M - 1).$$

This orthogonality property (see par. 10) is basic in the theory of finite Fourier series (or "trigonometric interpolation"), used for expanding functions defined at a finite number of equidistant mesh points; most of the elegant Fourier series theory (or more generally, orthogonal function theory) carries over with appropriate modifications in form, such as replacing infinite series and integrals by finite sums. We note finally that the solution of (2) with accessory conditions (13′) can be written in other forms (involving infinite series, integrals, etc.), but all forms give the same values at the mesh points (i.e., the solution is *unique* on the mesh). It seems to us that the form (17) is the simplest and most elegant.

Returning to our numerical study of equation (1), we first compare D, given by (13), with the numerical solution based on equation (2). We have kept Δx fixed and let Δt vary to give different values of r. In Plate 2 the values of r are stable values. We make the following observations:

(i) At $r < \frac{1}{2}$ all solutions are quite close to the true solution, $r = 0.45$ being very nearly as good as $r = 0.1$ in spite of the fact that not only is r 4.5 times as large but also Δt is 4.5 times as large. The fact that the curve representing $r = 0.45$ is better, over a portion of the graph, than $r = 0.1$, and better, over the entire graph, than $r = 0.3$, is of no special significance since the total time interval is only one-tenth of a second. We would expect $r = 0.3$ to give results closer to the true than $r = 0.45$ and an inspection of Plate 2 indicates that this will happen if we extend the time interval. However, since all of the curves are close to D, for the sake of speed it behooves us to choose r close to the limiting stable value.

(ii) We get decidedly poorer results using $r = 0.5$.

(iii) The gradual way in which $N(r = 0.1)$ approaches D as t increases is due to the nature of the heat equation (i.e., both Δ and D go to 0 as t goes to ∞) and to the fact that the accumulation of round-off errors is so trivial that Δ and N are numerically equivalent (see Table I, column 5).

The data from which Plate 2 is plotted are given in Table II, columns 2, 4, 5, and 6, and Table III, column 2.

In Plate 3 we compare D with numerical solutions, as above, using $r = 0.45$, 0.5, and the unstable values 0.55 and 0.7.[8] We note that:

(i) A value of r close to, but less than, 0.5 gives excellent results for this mesh size.

(ii) When r is chosen even slightly larger than 0.513, e.g. $r = 0.55$, the results oscillate with rapidly increasing amplitude.

(iii) A further slight increase in the value of r, e.g. $r = 0.7$, causes a tremendous increase in the amplitude of the oscillations.

The data for Plate 3 are found in Table II, column 2, 6, and 7, and Table III.

[8] Because not all error frequencies $|\beta|$ can be present, the largest r for which stability is guaranteed is 0.513. However, in accordance with our convention (footnote to par. 11) we call 0.5 the largest stable mesh-ratio since $r \leq 0.5$ guarantees nongrowth of an error for *any* frequency.

Let us now consider the results obtained by fixing r and varying the mesh size. We choose 0.1 and 0.3 as the values for r and find that, in all cases and in a regular manner, a finer mesh gives results very much closer to the solution of the differential equation; that is, we have *convergence* as well as *stability* (see Plates 4 and 5; Table II, columns 4 and 5, and Table IV). We note also that the curves, after a relatively large early separation, gradually get closer to the true solution D. This has already begun to show up in the last column of Table I. Since we are dealing with a bar that is gradually cooling to zero degrees this fact is not totally unexpected in spite of the accumulation of round-off errors.

Thus far we have compared N with D. We will now include in our comparison the *exact* solution of th difference equation, Δ, given by formula (17). Consider first, the numerical solution with $r = 0.1, \Delta x = 0.1$ (Table II, column 4) and the exact difference solution at $r = 0.1$ (Table V, column 2). The remarkable agreement indicates that, not only does a numerical error damp out when the mesh-ratio is 0.1, but also that the damping out is so rapid, that, coupled with the probability of many cancellations, the step-by-step accumulation of errors in the algorithm is very small. This experimentally supports the assumption that weak stability implies strong stability (see par. 6).

Finally, we consider another interesting question: how would the numerical solution N compare with the exact solution Δ of the difference equation if the largest stable value (i.e., $r = 0.5$) or an unstable value (e.g., $r = 0.55$) were used? We might expect, especially in the unstable case, that N would diverge quickly from Δ; such is not the result. The values of N are so nearly equal to the corresponding values of Δ that we have been able to join the Δ-points obtaining envelopes to the N-curves (as seen in Plate 6).[9] The brief tabulation below, taken from Tables II, III, and V, show at a glance the remarkable similarity between Δ and N in the unstable as well as the stable case.

$t(ms)$	$r = .5$ Δ	N	$t(ms)$	$r = .55$ Δ	N
20	0.9375	0.9375	22	0.9083	0.9085
30	0.8594	0.8594	38.5	0.8824	0.8824
45	0.78126	0.78125	55	0.5756	0.5753
60	0.6409	0.6409	71.5	0.7609	0.7609
85	0.5245	0.5245	88	0.2287	0.2287
100	0.4292	0.4292	104.5	0.8218	0.8218

The most important fact that the previous discussion brings out is that the error in the numerical solution (D-N), even in the unstable case, is not due to accumulation of numerical errors but almost entirely due to the fact that Δ diverges from D as was remarked in par. 5. This is also clear as regards a numeri-

[9] Because of the quantity $(1 - 2.2 \sin^2(\pi m/2M)) M^2 t/r$ in equation (17) we get a point on the curve for all values of t for which $M^2 t/r$ is an integer. The odd integers give a series of points which give the lower envelope and the even integers give points from which we draw the upper envelope.

cal solution of Richardson's difference equation (11); see Table I, columns 6 and 7. Here the exact solution of (11) is:

$$\Delta_E(x,t) = \sin \pi x/10\,[(1.26275)(.990259)^k + (-1)^k(.000000128073)(1.009837)^k]$$
$$+ \sin 3\pi x/10\,[(.392497)(.920950)^k + (-1)^k(.0000251924)(1.085836)^k]$$
$$+ \sin 5\pi x/10\,[(.199841)(.819804)^k + (-1)^k(.000159060)(1.219804)^k]$$
$$+ \sin 7\pi x/10\,[(.101628)(.731653)^k + (-1)^k(.000276714)(1.366767)^k]$$
$$+ \sin 9\pi x/10\,[(.0315333)(.683225)^k + (-1)^k(.000143606)(1.463647)^k] \quad (18)$$

The derivation of this formula deserves some comment. The general solution of (11) satisfying the boundary conditions is (we use the most convenient form for nine equally-spaced interior points—see par. 10)

$$S = \sum_{l=1,2,\cdots}^{9} \sin (l\pi x/10)[P_l \xi_{1l}^k + Q_l \xi_{2l}^k],$$

where $0 < \xi_{1l} = -1/\xi_{2l} \leq 1$. The P_l and Q_l are determined from initial conditions. Noting that (11) is a difference equation of the second order, we need to know the solution S on the two lines $k = 0,1$. The values on $k = 0$ are given to us, but we *choose* values on $k = 1$ according to some reasonable scheme. Adopting Richardson's scheme, we arrive at (18) which, as we have seen, does not converge. While following Richardson, since we discuss his numerical results, we note that the values on $k = 1$ can be so chosen that all the Q's vanish (i.e., we can eliminate that part of S which grows); then

$$\Delta'_E = \sum_{l=1,2,\cdots}^{9} \sin (l\pi x/10)\; P_l \xi_{1l}^k$$

is the solution. We remark without giving proof here, that Δ'_E converges to D as $\Delta x, \Delta t \to 0$, for all $r \equiv \Delta t/\Delta x^2$. To solve (1) by seeking to find Δ'_E numerically would seem to have several advantages: no annoying restrictions on Δt (see par. 15) and probably rather rapid convergence (as measured by the truncation error for meshes of increasing fineness). However, because the numerical determination of Δ'_E is inherently unstable (the Q's arising from the errors being, in general, not zero) one of several special devices must be used to limit the error growth (see reference 9, where an explicit discussion is given of error growth during the stepping-ahead solution of the inherently unstable Laplace's differential equation, for which all convergent difference schemes are unstable).

It should be noted that nearly all the numerical results exhibited at $x = 0.4$ were also computed at $x = 0.3$ and $x = 0.5$ and are in the hands of the authors. Since nothing essentially different or unexpected occurred for these values, we have presented, consistently, only values at $x = 0.4$ cm.

In conclusion, let us briefly summarize the contents of this paper. It has been

pointed out that in order to be useful (i.e., give an answer close to the true answer), a finite difference scheme for solving a partial differential problem should be both convergent and stable, where *convergence* and *stability* are precisely defined.[10] We have presented von Neumann's method of stability analysis and pointed out the advantages of implicit difference representations. We have shown that the exact solution, Δ, of the approximating difference equation may differ considerably in its behavior from the exact solution, D, of the differential equation (initial and boundary conditions being the same). It is suggested that in many cases deviations observed between D and N (the numerical solution of the difference equation) are due primarily to this difference in behavior of D and Δ and not to the accumulation and growth of round-off errors throughout the numerical calculation, as is generally thought. There are difference representations which are divergent and unstable for any mesh-size and any mesh-ratio. We have chosen as our principal example to solve numerically the one-dimensional heat equation $\partial\varphi/\partial t = \partial^2\varphi/\partial x^2$ with simple initial and boundary conditions. We study the effect on $\Delta(x, t)$ and $N(x, t)$ of varying: the mesh-size Δx, Δt; the mesh-ratio $\Delta t/(\Delta x)^2$; and the difference representation. Our detailed calculations confirm theoretical statements regarding convergence and stability for this parabolic problem.

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[10] If the scheme is convergent but unstable, some device must be used to limit error growth. See paragraph 4 and reference 9.

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Table I *

t(ms)	D	Δ_R	N_R	$\Delta=N$	$D-\Delta_R$	Δ_R-N_R	$D-\Delta$
5	1.0000	.9990	.9990	.9996	.0010	.0000	.0004
10	.9953	.9872	.9870	.9919	.0081	.0002	.0034
15	.9785	.9941	.9957	.9729	-.0156	-.0016	.0056
20	.9518	.7491	.7390	.9452	.2027	.0101	.0066
25	.9192	2.3771	2.5504	.9123	-1.4579	-.1733	.0069
30	.8832	-9.7547	-10.8768	.8766	10.6379	1.1221	.0066

* In all the Tables and Plates, the following terminology is used:

D = exact solution of differential equation (1)

Δ = exact solution of difference equation (2)

N = numerical solution of difference equation (2)

Δ_R and N_R are, respectively, the exact and numerical solutions of Richardson's difference equation (11).

Table II

			$\Delta x = 0.1$			
t(ms)	D	R	N(r=0.1)	N(r=0.3)	N(r=0.5)	N(r=0.7)
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1		1.0000	1.0000			
2		1.0000	1.0000			
3		.9999	1.0000	1.0000		
4		.9994	0.9999			
5	1.0000	.9990	0.9996		1.0000	
6		.9975	0.9989	1.0000		
7		.9970	0.9979			1.0000
8		.9936	0.9964			
9		.9940	0.9944	1.0000		
10	0.9953	.9870	0.9919		1.0000	
11		.9909	0.9890			
12		.9763	0.9856	0.9919		
13		.9898	0.9818			
14		.9585	0.9775			1.0000
15	0.9785	.9957	0.9729	0.9789	1.0000	
16		.9267	0.9679			
17		1.0193	0.9627			
18		.8652	0.9571	0.9623		
19		1.0845	0.9513			
20	0.9518	.7390	0.9452		.9375	
21		1.2439	0.9389	0.9432		1.0000
25	0.9192	2.5504	0.9123		.9375	
27		4.5251	0.8982	0.9005		
28		-4.4469	0.8911			0.7599
30	0.8832	-10.8768	0.8766	0.8779	0.8594	
35	0.8461		0.8398		0.8594	1.1441
36			0.8324	0.8322		
40	0.8088		0.8029		0.7812	
42			0.7883	0.7869		0.1717
45	0.7721		0.7666	0.7647	0.7812	
48			0.7453	0.7430		
49			0.7382			1.8908
50	0.7363		0.7312		0.7080	
54			0.7038	0.7009		

(Continued)

Table II (Continued)

t(ms)	D	R	N(r=0.1)	N(r=0.3)	N(r=0.5)	N(r=0.7)
55	0.7018		0.6971		0.7080	
56			0.6904			-1.4538
60	0.6686		0.6642	0.6608	0.6409	
63			0.6452	0.6416		3.4096
65	0.6368		0.6328		0.6409	
66			0.6266	0.6229		
70	0.6063		0.6027		0.5798	
72			0.5910	0.5870		
75	0.5773		0.5739	0.5698	0.5798	
78			0.5573	0.5531		
80	0.5496		0.5465		0.5245	
84			0.5254	0.5212		
85	0.5232		0.5203		0.5245	
90	0.4981		0.4954	0.4911	0.4745	
95	0.4741		0.4716		0.4745	
96			0.4670	0.4627		
100	0.4513		0.4490		0.4292	
102				0.4359		

Table III

t(ms)	N(r=0.45)	N(r=0.55)
0	1.0000	1.0000
4.5	1.0000	
5.5		1.0000
9.0	1.0000	
11.0		1.0000
13.5	1.0000	
16.5		1.0000
18.0	0.9590	
22.0		0.9085
22.5	0.9426	
27.0	0.9047	
27.5		0.9451
31.5	0.8743	
33.0		0.7976
36.0	0.8373	
38.5		0.8824
40.5	0.8043	
44.0		0.6856
45.0	0.7692	
49.5	0.7371	0.8273
54.0	0.7046	
55.0		0.5753
58.5	0.6744	
60.5		0.7857
63.0	0.6446	
66.0		0.4649
67.5	0.6166	
71.5		0.7609
72.0	0.5893	
76.5	0.5636	
77.0		0.3509
81.0	0.5386	
82.5		0.7560
85.5	0.5151	
88.0		0.2287
90.0	0.4923	
93.5		0.7746
94.5	0.4707	
99.0	0.4499	0.0928
103.5	0.4301	
104.5		0.8218
110.0		-0.0639
115.5		0.9042
121.0		-0.2502

Table IV

	r = 0.1		r = 0.3	
t(ms)	N(ΔX =.05)	N(ΔX =.2)	N(ΔX =.05)	N(ΔX =.2)
0	1.000000	1.000000	1.000000	1.000000
1.5	1.000000			
3.0	0.999997			
4.0	0.999970	1.000000		
4.5	0.999928			
6.0	0.999540		0.999934	
7.5	0.998469		0.999299	
8.0	0.997905	0.990000		
9.0	0.996420		0.997613	
10.5	0.993243		0.994666	
12.0	0.988919	0.973000	0.990450	1.000000
13.5	0.983508		0.985059	
15.0	0.977110		0.978621	
16.0	0.972354	0.951200		
16.5	0.969842		0.971275	
18.0	0.961820		0.963153	
19.5	0.953154		0.954373	
20.0	0.950140	0.926210		
21.0	0.943944		0.945041	
22.5	0.934278		0.935251	
24.0	0.924235	0.899205	0.925084	0.910000
28.0	0.896113	0.871039		
32.0	0.866812	0.842331		
36.0	0.837052	0.813525		0.811000
40.0	0.807323	0.784938		
44.0	0.777957	0.756792		
48.0	0.749174	0.729240		0.719200
50.0	0.735048			
52.0		0.702386		

Table V

t(ms)	Δ(r=0.1)	Δ(r=0.5)	Δ(r=0.55)
10.0	0.9919	1.0000	
11.0			1.0000
15.0		1.0000	
16.5			1.0000
20.0	0.9452	0.9375	
22.0			0.9083
25.0		0.9375	
27.5			0.9451
30.0	0.8766	0.8594	
33.0			0.7975
35.0		0.8594	
38.5			0.8824
40.0	0.8029	0.7812	
44.0			0.6856
45.0		0.7812	
49.5			0.8273
50.0	0.7312	0.7080	
55.0		0.7080	0.5756
60.0	0.6642	0.6409	
60.5			0.7857
65.0		0.6409	
66.0			0.4650
70.0	0.6026	0.5798	
71.5			0.7609
75.0		0.5798	
77.0			0.3509
80.0	0.5464	0.5245	
82.5			0.7560
85.0		0.5245	
88.0			0.2287
90.0	0.4954	0.4745	
93.5			0.7746
95.0		0.4745	
99.0			0.0928
100.0	0.4490	0.4292	
104.5			0.8218
105.0		0.4292	
110.0			-0.0640

PLATE 1

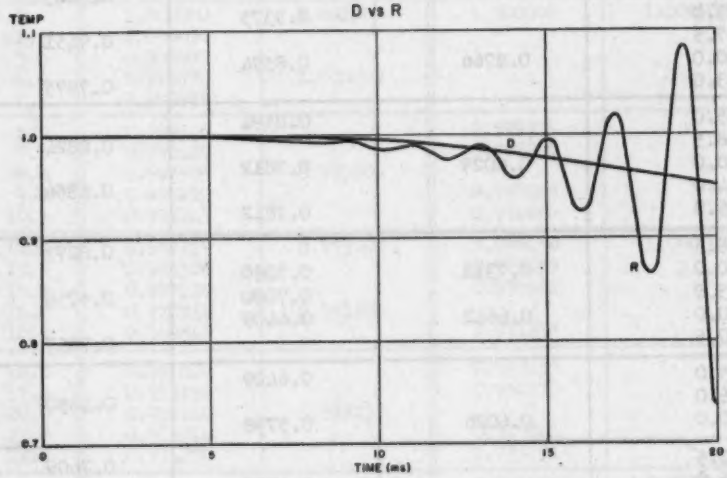


PLATE 2

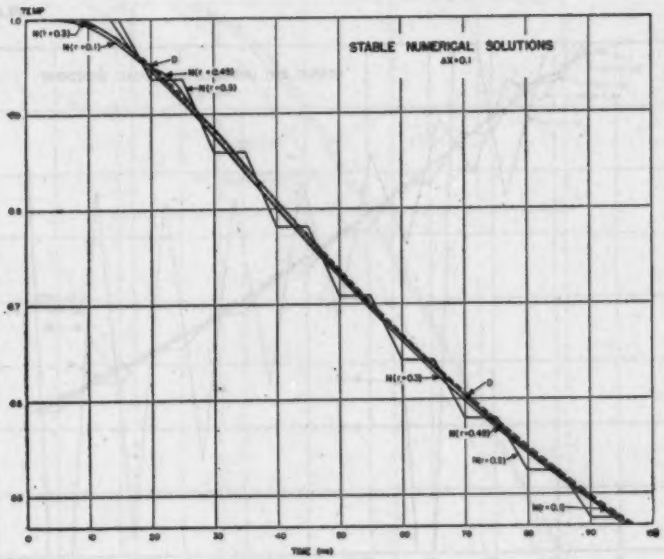


PLATE 3

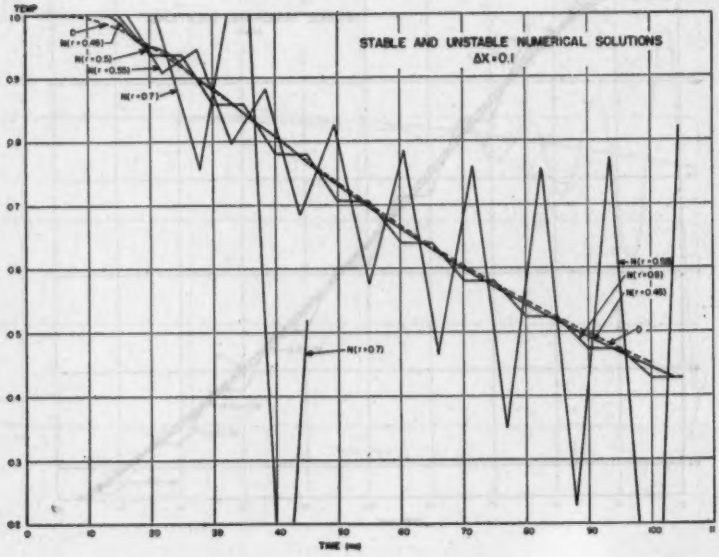

STABLE AND UNSTABLE NUMERICAL SOLUTIONS
ΔX = 0.1
TEMP
TIME (ms)
R(r = 0.7)
R(r = 0.5)
R(r = 0.48)

PLATE 4

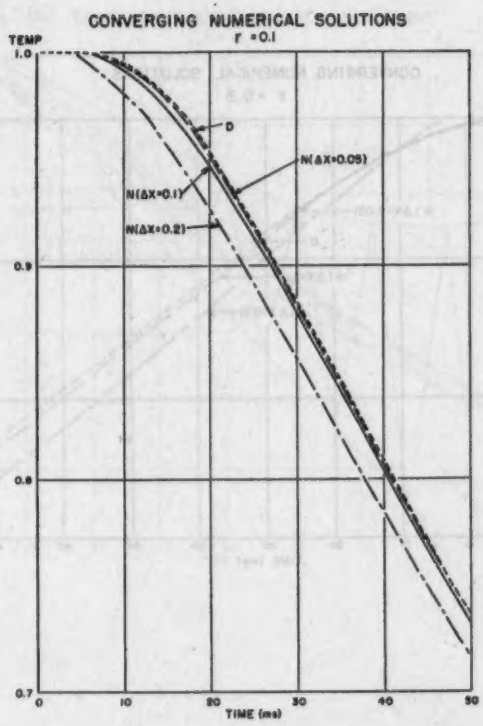

CONVERGING NUMERICAL SOLUTIONS
r = 0.1
TEMP
1.0
0.9
0.8
0.7
0
10
20
30
40
50
TIME (ms)
D
N(ΔX=0.05)
N(ΔX=0.1)
N(ΔX=0.2)

PLATE 6

CONVERGING NUMERICAL SOLUTIONS
r = 0.3

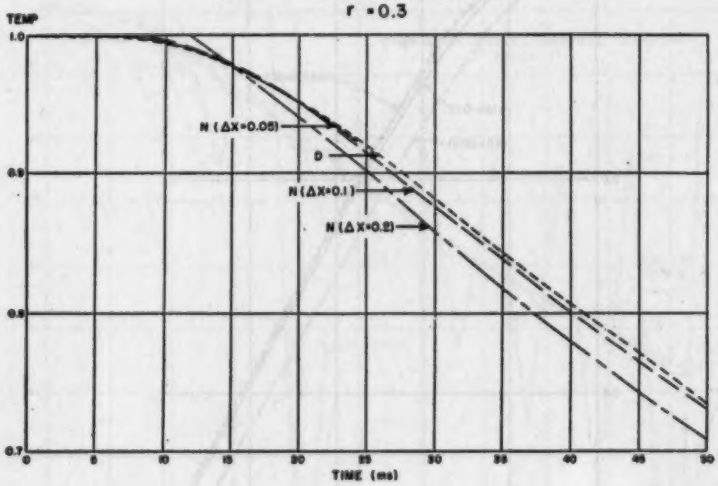
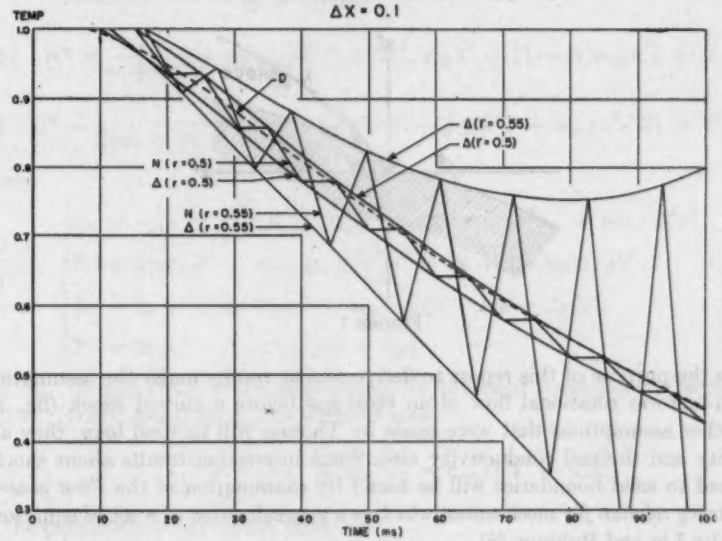


PLATE 6

NUMERICAL AND EXACT DIFFERENCE EQUATION SOLUTIONS
$\Delta X = 0.1$



THE GENERAL CONSISTENCY RELATIONS FOR SHOCK WAVES*

By W. F. Brown

1. Introduction. In a series of papers [1]–[4] T. Y. Thomas has derived a set of relations between the curvatures of the shock, the curvatures of the stream lines behind it, and their derivatives. These he has called *The consistency relations for shock waves*. All of this work was done under the assumption of uniform flow of an ideal gas before the shock.

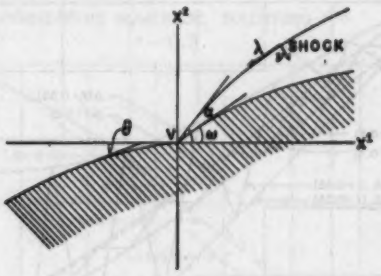


Figure 1

It is the purpose of this report to derive similar results under the assumption of non-uniform rotational flow of an ideal gas before a curved shock (fig. 1). The other assumptions that were made by Thomas will be used here; they are viscosity and thermal conductivity zero. Some interesting results about shocks attached to solid boundaries will be found by examination of the *First general consistency relation for shock waves*, which is a generalization of a set of equations found by Lin and Rubinov [5].

2. First derivatives behind the shock. Since considerable use will be made of some of the equations from our references, a new notation will now be introduced to simplify our work. The symbol $[i(j)]$ will mean equation j from reference i, e.g., [1(2)] denotes equation 2 from [1]. Whenever we use an equation from another paper, it will be clear that it is applicable to non-uniform rotational flow of an ideal gas.

In the following work we shall denote by q_1, $u_{1\alpha}$, ρ_1, p_1 and M, the velocity, velocity components, density, pressure and Mach number of the flow in front of the shock and correspondingly by q, u_α, ρ, p and M_2, the values of the quantities behind the shock.

In [1] the first derivatives behind the shock are found in terms of certain invariants $A_{,j}$ and a_i. These invariants are given by equations [1(12)], [1(16)] and certain relations, namely, $a_1 = C^*$; $a_2 = -\rho u_{\sigma,\sigma}$. The explicit form of A^*_α, B^* and C^* are needed in equations [1(12)], [1(16)] and $a_1 = C^*$; we find

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these from equations [1(7)] $\cdots$ [1(9)]. Since all of these results are applicable to non-uniform rotational flow, we shall now make use of them with a slight notational change. We shall denote the invariants given by equations [1(12)] and [1(16)] by $\bar{A}_{ij}$ and the other invariants by $\bar{a}_i$.

Now following the above procedure we first take the derivative of the right hand sides of equations [1(1)] $\cdots$ [1(3)] along the shock line. Then substitute these derivatives into equations [1(7)] $\cdots$ [1(9)] to get

$$(1) \qquad A_\alpha^* = \frac{1}{(\gamma+1)\rho_1 u_{1n}} \{L_\alpha E + F_\alpha\} + A_\alpha$$

$$(2) \qquad B^* = \frac{1}{\gamma+1} \{4\rho_1 u_{1n} u_{1\alpha,\beta}\lambda^\beta \nu^\alpha + 2u_{1n}^2 \rho_{1,\beta}\lambda^\beta + (1-\gamma)p_{1,\beta}\lambda^\beta\} + B$$

$$(3) \qquad C^* = \frac{1}{2\gamma p_1 + (\gamma-1)\rho_1 u_{1n}^2} \{\rho_{1,\alpha}\lambda^\alpha X + p_{1,\alpha}\lambda^\alpha Y + u_{1\alpha,\beta}\lambda^\beta \nu^\alpha Z\} + C$$

where

$$(4) \qquad \begin{cases} L_\alpha = -(\gamma+1)\,[u_\alpha] - 2u_{1n}\nu^\alpha; & E = \rho_{1,\beta}\lambda^\beta u_{1n} + \rho_1 u_{1\gamma,\beta}\lambda^\beta \nu^\gamma \\ F = 2(\gamma p_{1,\beta}\lambda^\beta - \rho_1 u_{1n} u_{1\gamma,\beta}\lambda^\beta \nu^\gamma)\nu^\alpha + (\gamma+1)\rho_1 u_{1n} u_{1\alpha,\beta}\lambda^\beta \\ X = (\gamma+1)\,(p+p_1) + 2\rho_1 - [(\gamma-1)\rho - 2\rho_1]u_{1n}^2 \\ Y = 2\gamma p; \qquad Z = u_{1n}(4\rho_1^2 - 2[\rho]) \end{cases}$$

and A_α, B and C are given respectively by the right hand sides of equations [1(19)]; [1(20)] and [1(21)]; also we have used

$$\frac{du_{1n}}{ds} = \frac{du_{1\alpha}}{ds}\nu^\alpha + u_{1\alpha}\frac{d\nu^\alpha}{ds} = u_{1\alpha,\beta}\lambda^\beta \nu^\alpha - u_t \kappa$$

in the derivation of the above relations.

It is to be noted that λ is the unit tangent vector to the shock line and that it is so directed that the perpendicular vectors λ, ν form a right handed system. Also s denotes arc length along the shock line, chosen to increase algebraically in the direction specified by the above vector λ. The curvature of the shock is denoted by κ. Also $u_{1n} = u_{1\alpha}\nu^\alpha$; $u_n = u_\alpha \nu^\alpha$ and $u_t = u_{1\alpha}\lambda^\alpha$.

By putting the above values of A_α^* and B^* into equations [1(12)] and [1(16)] the invariants $\bar{A}_{ij}$ are found to be

$$(5) \qquad \bar{A}_{11} = \frac{1}{(\gamma+1)\rho_1 u_{1n}} \{L_\alpha \lambda^\alpha E + F_\alpha \lambda^\alpha\} + A_{11}$$

$$(6) \qquad \bar{A}_{12} = -\frac{1}{(\gamma+1)\rho} \{4\rho_1 u_{1n} u_{1\alpha,\beta}\lambda^\beta \nu^\alpha + 2u_{1n}^2 \rho_{1,\beta}\lambda^\beta + (1-\gamma)p_{1,\beta}\lambda^\beta\} + A_{12}$$

$$(7) \qquad \bar{A}_{21} = \frac{1}{(\gamma+1)\rho_1 u_{1n}} \{L_\alpha u^\alpha E + F_\alpha u^\alpha\} + A_{21}$$

$$(8) \qquad \bar{A}_{22} = \frac{\gamma p}{\rho u_n^2 - \gamma p} \{-(\bar{A}_{12} + \bar{A}_{21})u_t + q^2 \bar{A}_{11}\} + A_{22}$$

where the A_{ij} are invariants for uniform flow, and hence are given by equations [1(25)] $\cdots$ [1(28)]. No attempt will be made to calculate $\bar{a}_i$, for it will be enough to note that these invariants have the same general form as the $\bar{A}_{ij}$ in their dependence on the derivatives of quantities in front of the shock line and on the corresponding invariants for uniform flow.

Our assumption of non-uniform rotational flow before the shock changes the invariants by adding a term which is homogeneous in first derivatives of quantities in front of the shock. These added parts will be called $\tilde{A}_{ij}$ and $\tilde{a}_i$ so that

$$\tilde{A}_{ij} = \bar{A}_{ij} - A_{ij}\,; \qquad \tilde{a}_i = \bar{a}_i - a_i\,.$$

Now the $\tilde{A}_{ij}$ can be written in a more compact fashion. To do this we find the first derivatives of quantities before the shock in terms of K^1 where K^1 is the curvature of the stream line of the incoming flow. For this certain derivatives will be needed. These will be gotten by simply replacing quantities behind the shock by quantities in front and ω by θ in the expressions for derivatives which have been found before [6]. After this change the derivatives in question are

$$(9)\quad \begin{cases} \dfrac{dp_1}{d\sigma} = -\rho_1 q_1 \dfrac{dq_1}{d\sigma}\,; \qquad \dfrac{d\rho_1}{d\sigma} = -\dfrac{\rho_1^2 q_1}{\gamma p_1}\dfrac{dq_1}{d\sigma}\,; \qquad (M^2-1)\dfrac{dp_1}{d\sigma} = -\rho_1 q_1^2 \dfrac{d\theta}{dn} \\ \dfrac{dp_1}{dn} = -\rho_1 q_1^2 K^1\,; \qquad \dfrac{dq_1}{dn} = q_1 K^1 + A p_1\,; \\ \dfrac{d\rho_1}{dn} = -\rho_1 M^2 K^1 + \left(\dfrac{\gamma-1}{\gamma}\right) A \rho_1^2 q_1 \end{cases}$$

where σ is the arc length along a stream line in the incoming flow, M is the Mach number of this flow, and Ap_1 is equal to the rotation of the flow; it can be shown that A is constant along stream lines [7]. Let us note that since θ is measured from the x^1 axis to the stream line in front of the shock, K^1 will be negative for the flow under consideration (see fig. 1).

Obviously the following derivatives along the shock may be expressed as

$$(10)\quad \begin{aligned} \frac{dp_1}{ds} &= -\left(\frac{dp_1}{dn}\sin\alpha + \frac{dp_1}{d\sigma}\cos\alpha\right), & \frac{d\rho_1}{ds} &= -\left(\frac{d\rho_1}{dn}\sin\alpha + \frac{d\rho_1}{d\sigma}\cos\alpha\right), \\ \frac{dq_1}{ds} &= -\left(\frac{dq_1}{dn}\sin\alpha + \frac{dq_1}{d\sigma}\cos\alpha\right), & \frac{d\theta}{ds} &= -\left(\frac{d\theta}{dn}\sin\alpha + \frac{d\theta}{d\sigma}\cos\alpha\right). \end{aligned}$$

From (10) with use of (9) and some manipulation we arrive at the result that

$$(11)\quad \frac{dp_1}{ds} = K^1\left\{\rho_1 q_1^2 \sin\alpha - \frac{dp_1}{d\theta}\cos\alpha\right\}$$

$$(12)\quad \frac{d\rho_1}{ds} = K^1\left\{\left[\rho_1 M^2 - \left(\frac{\gamma-1}{\gamma}\right)\rho_1 M^2 \Gamma\right]\sin\alpha - \frac{M^2}{q_1^2}\frac{dp_1}{d\theta}\cos\alpha\right\}$$

$$(13)\quad \frac{dq_1}{ds} = K^1\left\{-(1+\Gamma) q_1 \sin\alpha + \frac{1}{\rho_1 q_1}\frac{dp_1}{d\theta}\cos\alpha\right\}$$

$$(14)\quad \frac{d\theta_1}{ds} = K^1\left\{\frac{M^2-1}{\rho_1 q_1^2}\frac{dp_1}{d\theta}\sin\alpha - \cos\alpha\right\}$$

where we have taken

$$dp_1/d\sigma = (dp_1/d\theta)\,(d\theta/d\sigma) = (dp_1/d\theta)K^1$$

and have defined

$$\Gamma = Ap_1/q_1K^1. \tag{15}$$

Our definition of Γ tells us that it is zero when the incoming flow is irrotational and that it is dimensionless. An explicit form of A is found by solving [6(69)]. The result is

$$A = \left(\frac{dp_1}{dS}\right)^{-1}(\tau)^{-1}\left\{q_1 - \frac{(M^2-1)}{q_1^3}\left(\frac{1}{\rho_1}\frac{dp_1}{d\theta}\right)\right\}K^1$$

where S is the enthropy and τ is the gas constant. Replacing A in (15) by its above value, it is found that

$$\Gamma = \left(\frac{d}{dS}\ln p_1\right)^{-1}(\tau)^{-1}\left\{1 - (M^2-1)\left(\frac{1}{\rho_1 q^2}\frac{dp_1}{d\theta}\right)^2\right\}.$$

In our calculations we shall make use of the relations

$$du_{11}/ds = dq_1/ds; \qquad du_{12}/ds = q_1 d\theta/ds$$

which are valid only at the origin of our coordinate axes. Because our axes have been chosen so that the incoming flow is tangent to the x^1 axis at the vertex, the first of the above two relations follows immediately. The second is found by considering a set of coordinate axes in which the incoming flow makes an angle θ with the x^1 axis. Then it is seen that

$$du_{12}/ds = q_1 \cos\theta(d\theta/ds) + \sin\theta(dq_1/ds).$$

Putting $\theta = 0$ we get our result.

Using (8), (11) $\cdots$ (13) and the above relations we find, after some calculation, the invariants $\bar{A}_{ij}$ to be given by

$$\bar{A}_{11} = q_1\bar{B}_{11}K^1; \qquad \bar{A}_{21} = u_{1n}^2\bar{B}_{21}K^1$$
$$\bar{A}_{12} = qu_n\bar{B}_{12}K^1; \qquad \bar{A}_{22} = qu_tu_n\bar{B}_{22}K^1.$$

In these equations the $\bar{B}_{ij}$ are dimensionless expressions, which can be written in the form

$$\bar{B}_{ij} = [1/(\gamma+1)]\,\{C_{ij} + \Gamma D_{ij}\}.$$

The C_{ij} and D_{ij} are likewise dimensionless and are given by

$$C_{11} = -2ag - f\{1 + (M^2-2)a^2\}.$$

$$C_{12} = be[2(2\bar{a}g^2 - 2 - M^2) + fg(2e + 4(M^2-2) + (\gamma-1)\bar{a}^2)] + (\gamma-1)\bar{a}b$$

$$C_{21} = [\bar{a}^2\bar{M}^2\bar{e} - 6\bar{e}]\,[\bar{a}g^2 - a - gf(2-M^2)] + [(\gamma+1)\bar{a}^2 - 2\bar{a}^2\bar{e}\gamma]\,[a + gf]$$
$$+ M^2[\bar{a}^2\bar{M}^2\bar{e} - 4\bar{e}]\,[a - gf]$$

$$C_{22} = \bar{a}\bar{c}deC_{11} - dC_{12} - bde^2C_{21}$$

$$D_{11} = -ag; \qquad D_{12} = 2abe[(1 - \gamma^{-1})M^2 - 2]$$

$$D_{21} = \bar{a}(\gamma + 1) + [6\bar{e} - \bar{a}^2\bar{M}^2\bar{e}] + aM^2(\gamma - 1)[4\bar{e} - \bar{a}^2\bar{M}^2\bar{e}]$$

$$D_{22} = \bar{a}\bar{c}deD_{11} - dD_{12} - bde^2D_{21}$$

in which, to simplify our equations, we have defined $a \cdots h$ as follows

$$a = \frac{u_{1n}}{q_1} = -\sin\alpha; \qquad b = \frac{u_n}{v} = \sin(\alpha - \omega); \qquad c = \frac{u_t}{v} = -\cos(\alpha - \omega)$$

$$d = \frac{1}{(u_n^2/c^2) - 1} = \frac{hc^2}{M^2g^2be - hc^2}; \qquad f = \frac{1}{\rho_1 q_1^3}\frac{dp_1}{d\theta} = \frac{1}{\gamma M^2 p_1}\frac{dp_1}{d\theta}$$

$$e = \frac{\rho}{\rho_1} = \frac{2(M^2\sin^2\alpha - 1)}{(\gamma - 1)M^2\sin^2\alpha + 2} + 1; \qquad g = \cos\alpha$$

$$h = \frac{p}{p_1} = \frac{2\gamma}{\gamma + 1}(M^2\sin^2\alpha - 1) + 1$$

Also the reciprocals of $a \cdots h$, M are denoted by use of a bar, i.e., $\bar{a} = 1/a$, etc. As previously mentioned the $\tilde{a}_i$ shall not be calculated, but it is evident that they have the same form as the $\tilde{A}_{ij}$. From the above equations, we find that the $\tilde{A}_{ij}$ are proportional to K^1 with the factors of proportionality depending on α, $(d/d\theta)\ln p_1$, Γ and quantities in front of the shock. By using the value of A_{ij} from [1] and the above values for $\tilde{A}_{ij}$ it can be stated that

$$\tilde{A}_{ij} = \tilde{P}_{ij}K^1 + P_{ij}\kappa,$$

where the $\tilde{P}_{ij}$ are functions of α, $d\ln p_1/d\theta$, Γ, q_1, p_1, ρ_1 while the P_{ij} depend only on α, q_1, p_1, ρ_1.

If our flow in front is uniform then the above equations reduce to

$$\tilde{A}_{ij} = P_{ij}\kappa = A_{ij}$$

because $K^1 = 0$ for this flow.

The above expressions for $\tilde{A}_{ij}$ plus equations [1(14)], [1(17)] and [1(18)] tell us that the first derivatives behind the shock will be such that each will be composed of two terms, one homogeneous in κ, the other in K^1. The part homogeneous in κ is exactly the first derivative behind the shock for the uniform flow.

3. Relation between curvatures of shock and stream lines. In equation [1(34)], which is valid for non-uniform flows, we replace A_{12} and A_{22} by $\tilde{A}_{12}$ and $\tilde{A}_{22}$ respectively and get

$$K = -(\tilde{A}_{12}/vu_n) + (\tilde{A}_{22}/v^3u_n)$$

where v is the magnitude of the velocity vector behind the shock. Now putting the values of $\tilde{A}_{12}$ and $\tilde{A}_{22}$ found in section 2 into this equation, we obtain

$$K = G_0\kappa + U_0 \tag{16}$$

where G_0 is given in [1] and

$$U_0 = [1/(\gamma+1)]\{B_{12}(1-c^2d) - B_{21}b^2c^2de^2 + B_{11}bce^2(\gamma-1)\}K^1 \tag{17}$$

Equation (16) gives us the *first general consistency relation* which shall be examined more closely in another section.

For use in later work we note that U_0 can be written in the form

$$U_0 = \{F^{(1)}(M,\alpha)f + F^{(2)}(M,\alpha) - F^{(3)}(M,\alpha)\Gamma\}K^1$$

where the $F^{(i)}$ are dimensionless quantities. Since

$$U_0 = -(\tilde{A}_{12}/vu_n) + (\tilde{A}_{22}/v^2u_n)$$

it is zero when the flow is uniform. When the flow in front of the shock is uniform the first general consistency relation reduces to first consistency relations for uniform flows before a shock [2].

4. Explicit form of the derivatives of p. In this section the explicit form of the derivative of the pressure along a stream line behind the shock and also the derivative of the pressure along the shock shall be found. The arc length measured in the direction of the flow along the stream lines behind the shock shall be denoted by σ. For this we shall make use of the explicit form of du_2/ds. Obviously

$$du_2/ds = -[(du_2/dx^1)\sin\alpha + (du_2/dx^2)\cos\alpha].$$

To find the explicit form of the derivatives needed in the above equation, A_{ij} is replaced by $\tilde{A}_{ij}$ in [1(14)]. These values are then substituted into the above equation and then expanded; the final result is found to be

$$du_2/ds = q_1[G^{(1)}K^1 + G^{(2)}\kappa] \tag{18}$$

where

$$G^{(1)} = \tilde{B}_{11}b^2(a+g)\sin^2\omega - \tilde{B}_{21}[a^2eg/(\gamma+1)](\cos\omega - \sin\omega)$$

$$G^{(2)} = B_{11}ab^2\bar{e}(a+g)\sin^2\omega + B_{21}a^2b^{-2}c\bar{e}g(\cos\omega - \sin\omega)$$

and

$$B_{11} = \bar{e} - 1; \qquad B_{21} = (1-\bar{e})B_{11} + 4/(\gamma+1).$$

The derivation of the following relations

$$du_2/ds = q(d\omega/ds)$$

$$\frac{d\omega}{ds} = \left\{\left(\frac{M_2^2-1}{\rho q^2}\right)\frac{dp}{d\sigma}\sin\alpha - K\cos\alpha\right\}$$

holding behind the shock is similar to the derivation of the analogous relations which hold before the shock and which have been given above. Now from these two equations and equations (18) and (16) we get

$$dp/d\sigma = G^{(3)}K^1 + G^{(4)}\kappa \tag{19}$$

where

$$G^{(3)} = \left(\frac{M_2^2 - 1}{\rho q^2}\right)^{-1} [q_1 G^{(1)} \csc \alpha + qU_0 \cot \alpha]$$

$$G^{(4)} = \left(\frac{M_2^2 - 1}{\rho q^2}\right)^{-1} [q_1 G^{(2)} \csc \alpha + qG_0 \cot \alpha].$$

Now if we let $\alpha = \pi/2$ and $\kappa = 0$, equation (19) reduces to

$$\frac{dp}{d\sigma} \Big/ \frac{dp_1}{d\sigma} = -\frac{M^2 - 1}{1 - M_2^2} \tag{20}$$

which is the same result that has been found by Tsien [8]. By use of the explicit form of p/p_1, equation (20), and the following relation for normal shocks

$$1 - M_2^2 = \frac{(\gamma + 1)(M^2 - 1)}{2\gamma M^2 - (\gamma - 1)},$$

it is seen that we may write

$$\frac{dp}{d\sigma} \Big/ \frac{dp_1}{d\sigma} = -(p/p_1), \qquad (\alpha = \pi/2, \quad \kappa = 0).$$

The derivative of p_2 along the shock is found very simply, for obviously

$$dp/ds = p_{,\alpha}\lambda^{\alpha}.$$

The explicit form of $p_{,\alpha}$ is gotten by replacing A_{i2} by $\bar{A}_{i2}$ in equation [1(18)]. We thus find

$$dp/ds = -\rho \bar{A}_{i2} \zeta^1_{\alpha} \xi^{\alpha}_i = -\rho \bar{A}_{12}. \tag{21}$$

By using the previously found form of $\bar{A}_{12}$, we obtain

$$dp/ds = -\rho\{qu_n \bar{B}_{12} K^1 + u_n u_t B_{12} \kappa\}$$

where $B_{12} = 4/(\gamma + 1)$.

5. Extension to derivatives of high order. To derive the general consistency relations we shall, in the main, follow the method used in [2]. This involved first finding the second derivatives behind the shock from which $K_{(1)}$ is then found with $K_{(1)}$ defined to be $dK/d\sigma$; and finally, using the fact that this method was extensible to higher derivatives, $K_{(n)}$ was found.

In our work we shall need the form of dA_{ij}/ds and $d\bar{a}_1/ds$. First of all it is seen that

$$\frac{d\bar{A}_{ij}}{ds} = \frac{d\tilde{A}_{ij}}{ds} + \frac{dA_{ij}}{ds}; \qquad \frac{d\bar{a}_i}{ds} = \frac{d\tilde{a}_i}{ds} + \frac{da_i}{ds}.$$

A form of $\tilde{A}_{ij}$ which will be useful is

$$\tilde{A}_{ij} = I^{(1)}R + I^{(2)}K^1 + I^{(3)}N$$

where the $I^{(i)}$ in this equation are functions of α and quantities in front of the shock; also we have taken

$$R = R_{(0)} = d \ln p_1/d\sigma; \qquad N = N_{(0)} = Ap_1/q_1 .$$

For later use we put

$$R_{(n)} = (d^n/ds^n)\ (d \ln p_1/d\sigma)$$
$$N_{(n)} = (d^n/ds^n)\ (Ap_1/q_1)$$
$$K^1_{(n)} = (d^n/ds^n)\ (K^1).$$

Looking at $\tilde{A}_{ij}$ in the above equation it is seen that the derivatives of $\tilde{A}_{ij}$ and $\tilde{a}$ will be homogeneous polynomials in K^1, $K^1_{(1)}$, κ, R, $R_{(1)}$, N and $N_{(1)}$ with coefficient depending on quantities in front but not their derivatives. Now we must examine dA_{ij}/ds and da_i/ds; these break up into one part which is the same as that for the uniform flow before the shock, and another which is a homogeneous polynomial in K^1, κ, R and N. It is finally seen that $d\tilde{A}_{ij}/ds$ and $d\tilde{a}_i/ds$ consists of two terms, one the same as the derivatives of the invariants under the assumption of uniform flow before the shock. The other term is a homogeneous polynomial in K^1, $K_{(1)}$, κ, R, $R_{(1)}$, N and $N_{(1)}$.

We shall replace A_{ij} and a_i and their derivatives by $\tilde{A}_{ij}$ and $\tilde{a}_i$ and their derivatives, in the equations from [2]. This process is obviously legitimate.

Now the second derivatives are found by solving the system of linear equations given by [2(26)] $\cdots$ [2(28)] and [2(32)] $\cdots$ [2(34)]. Notice that in these equations the coefficients of the unknowns, the second derivatives, are not changed by using the assumption of non-uniform incoming flow. But the other terms are changed, since they contain first derivatives of quantities behind the shock, and derivatives of the invariants. To find how they are changed we look at [2(47)], [2(48)], [2(61)] $\cdots$ [2(63)] and reconsider our discussion of $d\tilde{A}_{ij}/ds$ and $d\tilde{a}_i/ds$. It is now evident that the terms which are changed are changed simply by adding onto their values for the uniform flow in front of the shock a homogeneous polynomial in K^1, $K^1_{(1)}$, κ, R, $R_{(1)}$, N and $N_{(1)}$. In consequence of this the second derivatives of quantities behind the shock are found to be the second derivatives for the uniform incoming flow plus a homogeneous polynomial in K^1, $K^1_{(1)}$, κ, R, $R_{(1)}$, N and $N_{(1)}$.

Now by taking the derivative of [1(33)] along any stream line behind the shock, we get

$$K_{(1)} = \frac{u_{\alpha,\beta\gamma} w_\alpha u_\beta u_\gamma}{v^4} + \frac{u_{\alpha,\beta} w_\alpha u_{\beta,\gamma} u_\gamma}{v^4} - \frac{3u_{\alpha,\beta} w_\alpha u_\beta u_{\gamma,\tau} u_\gamma u_\sigma}{v^6}.$$

From the form of the first and second derivatives, it follows that

$$K_{(1)} = G_{(1)}\kappa_{(1)} + H_1 + U_1 \tag{22}$$

where G_1 and H_1 are given in [2] and [6] respectively and U_1 is a homogeneous polynomial in K^1, $K^1_{(1)}$, κ, R, $R_{(1)}$, N and $N_{(1)}$ with dimensionless coefficients depending on M and α. This equation is our second general consistency relation.

To get our nth general consistency relation, we extend the above argument and obtain

$$K_{(n)} = G_n \kappa_{(n)} + H_n + U_n \tag{23}$$

where G_n is given in [4] and the general form of H_n in [2]. Here U_n is a homogeneous polynomial in $K^1, K^1_{(1)} \cdots K^1_{(n)}\ ; \kappa \cdots \kappa_{(n-1)}\ ; R \cdots R_{(n)}$ and $N \cdots N_{(n)}$ with dimensionless coefficients depending only on M and α. Note that at least one of the quantities $K^1 \cdots K^1_{(n)}\ ; R \cdots R_{(n)}$ or $N \cdots N_n$ occurs in each term of U_n.

When the flow is uniform in front of the shock, all the terms in U_n are zero, so equation (23) reduces to the one found previously for the uniform case.

To see that the coefficients in U_n depend only on M and α we use the following argument. Since $K_{(n)}$ has dimension $1/L^{n+1}$, U_n has dimension $1/L^{n+1}$. It is easily seen that $K^1_{(n)}\ ; \kappa_{(n)}\ , R_{(n)}$ and $N_{(n)}$ have dimension $1/L^{n+1}, n = 0, 1, 2 \cdots$. Now from the way U_n was derived the coefficients in this polynomial have zero dimension. Because of equations [1(1)] $\cdots$ [1(3)], we see that these dimensionless quantities are rational functions $t(q_1\ , p_1\ , \rho_1\ , \alpha)$. We know that $\rho_1 = \gamma M^2 p_1/q_1$. Put this value in t and get $t(p_1\ , M^2, \alpha)$. Since M^2 and α have no dimension, t must not depend on p_1, if it is to have zero dimension.

6. Conditions for normal shocks. In this section some interesting results will be derived from (16), for shocks attached to a solid boundary with continuously turning tangent, but possibly a discontinuity in curvature (fig. 2).

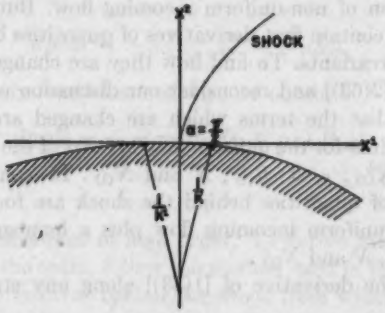

FIGURE 2

It is known in this case that $\alpha = \pi/2$, [9]. Replacing α by this value in (16) we get

$$\frac{K}{K^1} = \frac{2M^2}{(\gamma-1)M^2+2}\left\{M^2 - \frac{\gamma+3}{2}\right\} - \Gamma\{2(\gamma-1)M^2 - 4\}\left\{\frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}\right\}. \tag{24}$$

This shows that for a given Mach number, the ratio of the curvatures must have a definitely determined value, in order that a normal shock be formed. When $K = K^1$, which is the case of the most importance physically, the relation (24) becomes

$$1 = \frac{2M^2}{(\gamma-1)M^2+2}\left\{M^2 - \frac{\gamma+3}{2}\right\} - \Gamma\{2(\gamma-1)M^2-4\}\left\{\frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}\right\}. \tag{25}$$

If the flow in front is irrotational, then $\Gamma = 0$ and equation (24) reduces to

$$K = \frac{2M^2}{(\gamma-1)M^2+2}\left\{M^2 - \frac{\gamma+3}{2}\right\}K^1.$$

A number of other authors [5] and [8] have gotten this result by different methods, but the above method is considered to be more general.

Let us now investigate the case of a normal shock where κ becomes infinite. Now if $\alpha \to \pi/2$, we see from the graphs of G_0 in [9] that $G_0 \to 0^+$. So if $\alpha \to \pi/2$ and $\kappa \to \infty$ then $G_0\kappa \geqq 0$. If in equation (16) $\alpha \to \pi/2$ and $\kappa \to \infty$ then equation (16) reduces to the following relation

$$\frac{2M^2}{(\gamma-1)M^2+2}\left\{M^2 - \frac{\gamma+3}{2}\right\}K^1 + \Gamma\{2(\gamma-1)M^2-4\} \cdot \left\{\frac{(\gamma-1)M^2}{(\gamma-1)M^2+2}\right\}K^1 - K \leqq 0. \tag{26}$$

When the incoming flow is irrotational, i.e. $\Gamma = 0$, equation (26) reduces to

$$\frac{2M^2}{(\gamma-1)M^2+2}\left\{M^2 - \frac{\gamma+3}{2}\right\}K^1 - K \leqq 0.$$

Lin and Rubinov obtained this result by the use of a limit process [5].

7. Stability. In [3] it was shown that if α is in the shock region $[\beta(M) \leqq \alpha \leqq 90°]$ then the shock is unstable. $\beta(M)$ is the angle at which the Mach number behind the shock becomes unity. The method used in [3] is essentially the following. If α is in the region $[\beta(M) \leqq \alpha \leqq 90°]$ but all the consistency relations are satisfied, one makes a small change in the obstacle behind the shock; now since α is in the singular shock region $[\beta(M) < \alpha \leqq 90°]$, the new angle $\bar{\alpha}$ associated with the new obstacle will give us one of the $G_i = 0$. If the consistency relations are still satisfied, then by a further slight change in $\bar{K}_{(i)}$ the ith consistency relation will not be satisfied and no shock can exist. In this way the shock was said to be unstable.

The General Consistency Relations differ from the consistency relations for uniform flow only by U_n. Obviously U_n will not affect the argument used in [3]. So by applying the above argument from [3] to the General Consistency Relations

we see that if α is in the shock region $[\beta(M) \leqq \alpha \leqq 90°]$ for non-uniform rotational flow, the associated shock will be unstable.

In [9] it has been shown that we must associate only $\alpha = \pi/2$ with $\omega = 0$. This was done by using only the shock conditions given by equations [9(2)] $\cdots$ [9(4)], and the fact that the shock is a compression shock, i.e. $\rho/\rho_1 > 1$. Under our assumptions, i.e. non-uniform rotational flow of an ideal gas with viscosity and thermal conductivity zero, the shock conditions are still given by equation [9(2)] $\cdots$ [9(4)]; also we shall limit our attention to compression shocks. Hence we must associate only $\alpha = \pi/2$ with $\omega = 0$. In other words if, in a non-uniform flow of an ideal gas with viscosity and thermal conductivity zero, a shock is attached to a solid boundary with continuously turning tangent, it is a normal shock. Now from the discussion of instability of shocks, we see that since $\alpha = \pi/2$ is in the singular shock region, the normal shocks here considered are unstable, This is not to be interpreted as meaning that normal shocks can not occur, since such in fact has not been proved, but merely that their occurrence is unlikely when extraneous conditions are not present which tend to exert a stabilizing influence. Such conditions may be expected to arise, e.g., if the other extremity of the shock line is attached to a solid boundary at which the shock angle does not lie in the singular interval, or in the case of the reflection of shocks at the sonic line in transonic flow. Also under certain circumstances it may be impossible to neglect the effect of the boundary layer in consequence of which influences of a stabilizing nature may be introduced. Actually normal and even oblique shocks have been observed in flows around curved boundaries with continuously turning tangent, which appears to indicate the influence on the shock of the above or other factors outside the immediate neighborhood of the boundary [10].

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STEADY, PLANE, ROTATIONAL PRANDTL-MEYER FLOW OF A POLYTROPIC GAS*

By M. H. Martin

Introduction. It is well known that in classical Prandtl-Meyer corner flow of a polytropic gas the hodograph of the flow is an arc of an epicycloid.[1] If the maximum speed be taken as unity, the epicycloid is generated by rolling a circle of radius $s = (1 - c)/2$ upon the sonic circle of radius $r = c$, where c is the critical speed. The flow is isoenergetic, isentropic and therefore irrotational.

In this paper we investigate isoenergetic, Prandtl-Meyer flows of a polytropic gas which are not isentropic (i.e., *anisentropic* flows), a *Prandtl-Meyer flow* being, by definition, a flow in the physical plane whose hodograph is a single arc in the hodograph plane.

Many properties of isentropic flows are not shared by anisentropic flows. For isentropic flows $p = k\rho^\gamma$, and k is constant over the field of flow. Consequently in isentropic flows the isobars (p = const.) and isopycnics (ρ = const.) coincide. As a glance at Figures 7 and 8 in this paper verifies, this need not be the case for anisentropic flows. In addition, for isoenergetic, isentropic flow, the density always remains between finite bounds, since it is never negative and cannot exceed the stagnation density.

$$\rho_0 = [C(\gamma - 1)/k\gamma]^{1/(\gamma-1)}, \qquad k = \text{const. } C = \text{Bernoulli constant.}$$

In anisentropic flows $k = k(\psi)$ and it is possible for k to tend to zero as ψ varies. Consequently we can no longer always expect a finite density over a field of isoenergetic, anisentropic flow, and this is confirmed by the examples at the end of the paper.

In an earlier paper[2] it is shown that the distribution of entropy $S(\psi)$ from streamline to stream line cannot be taken arbitrarily for Prandtl-Meyer flows of a polytropic gas, but is restricted to two special forms, corresponding to which the entropic equation of state becomes respectively

$$p = A(\psi - \psi_0)^{\gamma/\lambda}\rho^\gamma, \qquad p = e^{A(\psi-\psi_0)}\rho^\gamma, \qquad (A = \text{const.}),$$

the second form replacing the first when $\lambda = 0$. The place of the epicycloid in the hodograph plane in classical theory is taken by a family of roulettes depending on the parameter λ. These roulettes are all generated by a point lying at a distance t from the center of a circle of radius s as this circle rolls externally or internally upon a circle of radius r. Here r, s, t are certain functions of λ defined in the text by (29), (19). Representative members of this family are shown in Figure 3.

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[1] See, for example, Courant, R., and Friedricks, K. O., *Supersonic Flow and Shock Waves*, Interscience Publishers (1948), pp. 250–278.

[2] Martin, M. H., *Steady, plane, rotational Prandtl-Meyer flows*. Naval Ordnance Laboratory Memorandum 10346, and J. of Math. and Phys. vol. 29 (1950) pp. 76–89.

In the physical plane, we meet rotational flows first discovered by Prim,[3] and the latter part of the paper is devoted to a detailed examination of the form of the stream lines, isobars, and isopycnics, as well as their behavior as the values of the parameters ψ, p, ρ are varied.

1. Fundamental principles. A flow is said to be *general* in a region of the physical plane provided in this region: (a) the isobars and streamlines do not coincide, (b) the isobars and isoclines (θ = const.) do not coincide, and (c) the speed q, the density ρ, and the pressure gradient dp/ds along a stream line remain finite and do not vanish. The problem of determining general flows can be reduced[4] to finding two functions

$$q(p, \psi) \equiv \textit{the Bernoulli function}, \quad \theta(p, \psi) \equiv \text{the } \textit{direction function},$$

which satisfy the equation

$$q[(q_{pp} - q\theta_p^2)/\theta_\psi]_\psi + (q^2\theta_p)_p = 0, \tag{1}$$

where the subscripts denote partial differentations. Once the Bernoulli and direction functions have been determined subject to this equation, a general flow is presented[4] in the physical plane by

$$\begin{aligned} x &= \int \left\{ -\frac{q_{pp} - q\theta_p^2}{\theta_\psi} \cos\theta \, dp + (q \sin\theta)_p \, d\psi \right\}, \\ y &= \int \left\{ -\frac{q_{pp} - q\theta_p^2}{\theta_\psi} \sin\theta \, dp - (q \cos\theta)_p \, d\psi \right\}, \end{aligned} \tag{2}$$

and in the hodograph plane by

$$u = q \cos\theta, \qquad v = q \sin\theta. \tag{3}$$

For a polytropic gas Bernoulli's law states

$$\tfrac{1}{2}q^2 = C - [p^{1-n}/(1-n)] \exp[(S - S_0)/C_p], \quad 0 < n = 1/\gamma < 1,$$

so that the Bernoulli function is known as soon as the entropy distribution $S = S(\psi)$ and the distribution of energy $C = C(\psi)$ are prescribed. Equation (1) then becomes a quasi-linear partial differential equation for the determination of the unknown direction function $\theta(p, \psi)$. If $C(\psi) \equiv$ const., the flow is *isoenergetic*, otherwise *non-isoenergetic*. If $S(\psi) \equiv$ const., the flow is *isentropic*, otherwise *anisentropic*. In particular we can always realize $C = \frac{1}{2}$ for isoenergetic flows by proper choice of units, so that Bernoulli's law becomes

$$q^2 = 1 - [2p^{1-n}/(1-n)] \exp[(S - S_0)/C_p]. \tag{4}$$

With this choice of C the maximum speed on all stream lines is unity.

[3] Prim, R. C. *On a family of rotational gas flows*. Naval Ordnance Laboratory Memorandum 9369, October 1947. Prim gives the graphs of the stream lines found in Figures 7 and 8 of the present paper without, however, showing isobars, isopycnics, or the regions of subsonic and supersonic flow.

For general flows the vorticity equals[4] $\rho q q_\psi$, and consequently general anisentropic flows of a polytropic gas are necessarily rotational.

When the hodograph (3) of a flow degenerates into an arc in the hodograph plane the flow in the physical plane is a *Prandtl-Meyer flow*. A Prandtl-Meyer flow which is general in a region of the physical plane is called a *general Prandtl-Meyer flow*, or briefly a *GPM-flow*.

From now on we restrict ourselves to GPM-flows of a polytropic gas under the hypothesis that they are isoenergetic and anisentropic. As has been pointed out above, such flows are necessarily rotational and there is no loss in generality in taking the Bernoulli function defined by (4).

We begin with the following theorem.[5]

THEOREM 1. *For isoenergetic, anisentropic GPM-flows of a polytropic gas the hodograph curves h: $\theta = \theta(q)$ in the hodograph plane constitute the solutions of the second order differential equation*

$$\theta'' - a(q, \lambda)\theta' - b(q, \lambda)\theta'^3 = 0, \qquad ' = d/dq, \tag{5}$$

in which

$$a = 2\lambda(q - q^{-1})^{-1} - q^{-3}[\tfrac{1}{2}(1 + q^{-2}) - l]^{-1},$$

$$b = [\tfrac{1}{2}(q - q^{-1}) - (\lambda + l)q]\,[\tfrac{1}{2}(1 + q^{-2}) - l]^{-1},$$

where $l = \gamma(\gamma - 1)^{-1}$, and λ is a real parameter which serves to determine the entropy distribution function $S(\psi)$ as follows:

$$\lambda \neq 0\text{:}\quad S(\psi) = (C_p/\lambda)\log(\psi - \psi_0) + \text{const.}; \qquad \lambda = 0\text{:}\quad S(\psi) = A(\psi - \psi_0)$$

Here C_p is the specific heat at constant pressure and ψ_0, $A \neq 0$ are any real constants.

Corresponding to the two forms of the entropy distribution function, the entropic equation of state reduces to the forms given in the introduction.

Differential equation (5) results[5] when a solution to (1) is sought of the form $\theta = \theta(q)$ for q defined by (4). This differential equation reduces to one of first order of Bernoulli type

$$\eta' + 2\left[\frac{\lambda q}{1 - q^2} - \frac{c^2}{q(q^2 - c^2)}\right]\eta = q\,\frac{(2\lambda c^2 + 1)q^2 + c^2}{q^2 - c^2}\,\eta^3 \tag{6}$$

when we set

$$\theta' = \eta, \qquad l = \tfrac{1}{2}(1 + c^{-2}), \tag{7}$$

where c, defined by

$$c^2 = (\gamma - 1)/(\gamma + 1)$$

is the *critical speed*.

[4] Martin, M. H. *Steady, rotational, plane flow of a gas*, Naval Ordnance Laboratoyr Memorandum 10274, and Am. J. of Math. vol. 72 (1950) pp. 465–484.

[5] Loc. cit. 2.

2. The hodograph curves h. The substitution

$$\eta = \alpha^{-1}, \tag{8}$$

carries (6) into a linear differential equation

$$\alpha' - 4\left[\frac{\lambda q}{1-q^2} - \frac{c^2}{q(q^2-c^2)}\right]\alpha + 2q\,\frac{(2\lambda c^2+1)q^2+c^2}{q^2-c^2} = 0. \tag{9}$$

As may be checked by direct substitution, the general solution of (9) is

$$\alpha = \frac{c^4}{\Lambda}\left(\frac{q^2(1-q^2)(q^2-\Lambda)}{(q^2-c^2)^2} + K\,\frac{q^4}{(1-q^2)^{2\lambda}(q^2-c^2)^2}\right), \tag{10}$$

(where K is an integration constant), provided

$$\Lambda = [(2\lambda+1)/(2\lambda c^2+1)]c^4, \tag{11}$$

is finite and not zero, that is, provided

$$\lambda \neq -\tfrac{1}{2}c^{-2}, \qquad \lambda \neq -\tfrac{1}{2}. \tag{12}$$

Letting $\Lambda \to \infty$ in (10) we are led to expect that

$$\alpha = -c^4\,\frac{q^2(1-q^2)}{(q^2-c^2)^2} + K\,\frac{q^4(1-q^2)^{1/c^2}}{(q^2-c^2)^2}, \tag{13}$$

is the general solution when $\lambda = -\frac{1}{2}c^{-2}$, as is indeed verified by substitution in (9).

For $\lambda = -\frac{1}{2}$ the general solution is

$$\alpha = \frac{q^2(1-q^2)[(1-c^2)q^2\log(1-q^2) - c^4]}{(q^2-c^2)^2} + K\,\frac{q^4(1-q^2)}{(q^2-c^2)^2}. \tag{14}$$

In view of (7), (8), and (12), (13), (14), the differential equation of the hodograph curves takes the following forms:

$$\theta'^2 = \frac{\Lambda}{c^4}\,\frac{(q^2-c^2)^2}{q^2(1-q^2)(q^2-\Lambda) + Kq^4(1-q^2)^{-2\lambda}}, \qquad \lambda \neq -\tfrac{1}{2}c^{-2}, \lambda \neq -\tfrac{1}{2}, \tag{15}$$

$$\theta'^2 = \frac{(q^2-c^2)^2}{-c^4q^2(1-q^2) + Kq^4(1-q^2)^{1/c^2}}, \qquad \lambda = -\tfrac{1}{2}c^{-2} \tag{16}$$

$$\theta'^2 = \frac{(q^2-c^2)^2}{q^2(1-q^2)[(1-c^2)q^2\log(1-q^2) - c^4] + Kq^4(1-q^2)}, \qquad \lambda = -\tfrac{1}{2}. \tag{17}$$

Throughout the paper our attention is directed to (15), with $K = 0$, except at the conclusion of section 3 where we touch briefly on the case $\lambda = -l, K > 0$.

If $K = 0$, $\Lambda = q_0^2 > 0$, the hodograph curves are obtained, from (15), by integrating

$$\theta'^2 = (q_0^2/c^4)\,[(q^2-c^2)^2/q^2(1-q^2)\,(q^2-q_0^2)]. \tag{18}$$

There is obviously no loss in generality in assuming $q_0 > 0$. Since the maximum speed is unity, (18) implies $0 < q_0 \leqq q \leqq 1$, and the hodograph curves are

accordingly confined to the annular ring between the circles $q = q_0$, $q = 1$. If we set $\Lambda = q_0^2$ in (11), the minimum speed q_0 is related to the parameter λ by

$$\lambda = (c^4 - q_0^2)/2c^2(q_0^2 - c^2). \tag{19}$$

It is clear from this that $0 < q_0 < 1$, if, and only if, λ is excluded from the interval

$$-l \leqq \lambda \leqq -\tfrac{1}{2}, \tag{20}$$

where l is defined in Theorem 1.

Direct insight into the nature of the hodograph curves is afforded by the following lemma.

LEMMA 1. *The locus of a point $qe^{i\theta}$ in the hodograph plane fixed upon a radius (prolonged, if necessary) at a distance $t > 0$ from the center of a circle of radius $|s|$ as this circle rolls upon another circle of radius $r > 0$ with center at the origin of the hodograph plane is a solution of the differential equation*

$$\theta'^2 = \left(\frac{r+2s}{r}\right)^2 \frac{[q^2 - \{[(r+s)^2 - t^2]/(r+2s)\}r]^2}{q^2[(r+s+t)^2 - q^2][q^2 - (r+s-t)^2]}, \tag{21}$$

the circle of radius $|s|$ rolling externally if $s > 0$, and internally if $s < 0$, upon the circle of radius r. For $s = 0$, the locus is a circle of radius t with center on the circle of radius r.

For the proof of this lemma we distinguish three cases corresponding to the three figures in Figure 1. First we treat cases (i), (iii) simultaneously and then take up case (ii).

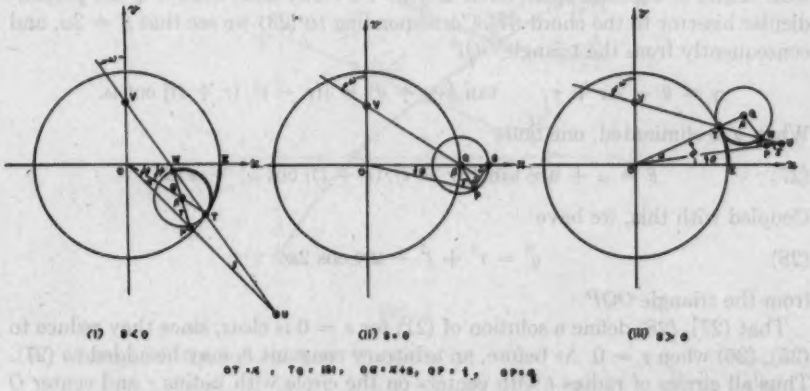

FIG. 1

The small circle with center Q is assumed to roll on the large circle with center O, the heavy arc TW of the smaller circle having made contact with the heavy arc ET of the larger circle so that

$$r\varphi = s\beta. \tag{22}$$

From triangle OTV we have $\varphi = \omega - \beta/2$ in both cases, and consequently

$$\beta = [2r/(r + 2s)]\omega, \tag{23}$$

in view of (22).

From triangle OUV we see that

$$\theta = \omega + \delta - \pi/2, \tag{24}$$

and from triangles PUW, OPQ it follows that $\delta = (\gamma - \alpha)/2$. The law of tangents applied to the triangle OPQ yields

$$\tan \delta = [(r + s - t)/(r + s + t)] \cot \beta/2.$$

It follows from (23), (24) that

$$\theta = \omega + \arctan ([(r + s - t)/(r + s + t)] \cot [r\omega/(r + 2s)]) - \pi/2, \tag{25}$$

where the branch between 0 and π is chosen for the arc tangent. From triangle OPQ

$$q^2 = (r + s)^2 + t^2 - 2t(r + s) \cos [2r\omega/(r + 2s)]. \tag{26}$$

Equations (25), (26) yield parametric equations for the locus of P as the small circle rolls on the larger and Lemma 1 is verified by substitution into (21). From (25) obviously $\omega \to 0$ implies $\theta \to 0$. By adding an arbitrary constant to (25) it is clear that we can realize $\theta \to \theta_0$ as $\omega \to 0$.

When $s = 0$, the locus of P is a circle of radius t with center Q on the circle with center 0, e.g., the small circle in (ii). To verify this, draw VQ the perpendicular bisector to the chord GP. Corresponding to (23) we see that $\beta = 2\omega$, and consequently from the triangle OQP

$$\gamma = \theta - 2\omega + \pi, \qquad \tan \tfrac{1}{2}(\gamma + \theta) = [(r - t)/(r + t)] \cot \omega.$$

When γ is eliminated, one finds

$$\theta = \omega + \arctan ([(r - t)/(r + t)] \cot \omega) - \pi/2. \tag{27}$$

Coupled with this, we have

$$q^2 = r^2 + t^2 - 2rt \cos 2\omega. \tag{28}$$

from the triangle OQP.

That (27), (28) define a solution of (21) for $s = 0$ is clear, since they reduce to (25), (26) when $s = 0$. As before, an arbitrary constant θ_0 may be added to (27). Thus all circles of radius t with centers on the circle with radius r and center O are solutions of (21) when $s = 0$.

If we place

$$r = \frac{1 + q_0}{q_0 + c^2} c^2, \qquad s = \frac{(1 + q_0)(q_0 - c^2)}{2(q_0 + c^2)}, \qquad t = \frac{1 - q_0}{2}, \tag{29}$$

in (21), we obtain (18) and the following theorem.

THEOREM 2. *For $K = 0$, $\Lambda = q_0^2 > 0$, the hodograph curves are generated by rolling a circle of radius $|s|$ on the circle $q = r$. The hodograph curves are traced*

out by a point $qe^{i\theta}$ fixed upon a radius of the rolling circle at a distance $t > 0$ from its center, provided r, s, t are defined by (29). *Their parametric equations in polar coordinates q, θ in terms of the angle ω in Figure* 1 *are*

$$q^2 = \sin^2 c^2\omega/q_0 + q_0^2 \cos^2 c^2\omega/q_0 ,$$
$$\theta = \omega + \arctan (q_0 \cot c^2\omega/q_0) - \pi/2 + \theta_0 \tag{30}$$

where θ_0 is an arbitrary constant and the branch between 0 *and* π *is chosen for the arctangent.*

The quantities r, s, t defined in (29) are, for fixed c, monotonic functions of q_0 in the interval $0 \leqq q_0 \leqq 1$ and their relative magnitudes for any given q_0 may be read off from the figure below, provided $c < 3c^2 < 1$. This assumption is met for most gases, e.g., $c^2 = 1/4$, $1/6$ for monatomic and diatomic gases respectively. In order to compare t and $|s|$ in the range $0 < q_0 < c^2$ we observe that

$$t + s = q_0(1 - c^2)/(q_0 + c^2) > 0,$$

and consequently

$$t > |s| \quad \text{for} \quad 0 < q_0 < c^2.$$

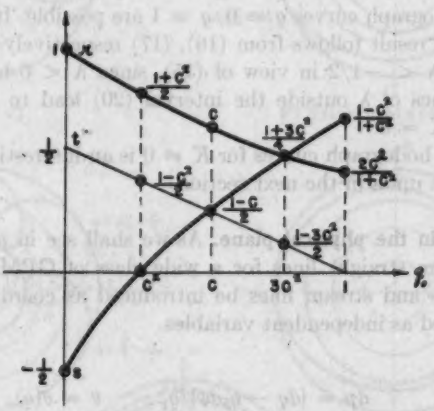

FIG. 2

We distinguish the following seven cases :

I: $0 < q_0 < c^2$, $\quad r > t > |s|$, $s < 0$;

II: $q_0 = c^2$, $\quad r > t > s = 0$;

III: $c^2 < q_0 < c$, $\quad r > t > s > 0$;

IV: $q_0 = c$, $\quad c = r > t = s = (1 - c)/2$,

V: $c < q_0 < 3c^2$, $\quad r > s > t > 0$,

VI: $q_0 = 3c^2$, $\quad r = s > t > 0$,

VII: $3c^2 < q_0 < 1$, $\quad s > r > t > 0$.

Seven different types of hodograph curves are obtained corresponding to these seven cases. They are shown in Figure 3. These figures indicate as (18) implies, that whenever a hodograph curve meets the sonic circle, it meets it at right angles.

From (18) we see that

$$\theta' = \pm(q_0/c^2)\,(q^2 - c^2)/q\sqrt{(1-q^2)(q^2-q_0^2)}, \tag{18'}$$

in which the positive sign is taken for the arcs marked $+$ in Figure 3 and the negative sign for the arcs marked $-$. If we set $\theta_0 = 0$ in (30) and let ω range from 0 to $\omega_m = \pi q_0/2c^2$ the $+$ arcs are traced from the point $\theta = 0$ on the circle $q = q_0$ to the point $\theta = \omega_m - \pi/2$ on the unit circle $q = 1$; if ω ranges from 0 to $-\omega_m$ the $-$ arcs are traced from the same point on the circle $q = q_0$ to the point $\theta = \pi/2 - \omega_m$ on the unit circle.

Case IV affords a check on our calculations, for, from (11), $q_0 = c$ corresponds to $\lambda = \infty$. In view of Theorem 1, this implies isentropic flow. Since the flow is isoenergetic it must be irrotational and it is well known that the hodographs of irrotational PM-flows of a polytropic gas are the epicycloids in IV of Figure 3.

This disposes of the case $K = 0$, provided λ does not lie in the interval (20). For $-l \leqq \lambda < -1/2c^2$ we have $\Lambda \geqq 1$ from (11), and therefore, by (15), (19), only trivial hodograph curves $q = 0$, $q = 1$ are possible. If $\lambda = -1/2c^2$ or $\lambda = -1/2$ the same result follows from (16), (17) respectively and remains in force for $-1/2c^2 < \lambda < -1/2$ in view of (15), since $\Lambda < 0$ holds because of (11). Thus only values of λ outside the interval (20) lead to hodograph curves of interest when $K = 0$.

The study of hodograph curves for $K \neq 0$ is an interesting problem which we shall only touch upon in the next section.

3. The flow in the physical plane. As we shall see in a moment the isovels (q = const.) are straight lines for a wide class of GPM-flows. This suggests that the isovels and stream lines be introduced as coordinate lines, i.e., that q, ψ be employed as independent variables.

If we set

$$dp = [dq - q_\psi d\psi]/q_p, \qquad \theta = \theta(q),$$

in (2) these equations become

$$x = \int\left\{\frac{qq_p^2\theta'^2 - q_{pp}}{q_p q_\psi \theta'}\cos\theta\,dq + \left(\frac{q_{pp}}{q_p^2\theta'}\cos\theta + \sin\theta\right)q_p\,d\psi\right\},$$

$$y = \int\left\{\frac{qq_p^2\theta'^2 - q_{pp}}{q_p q_\psi \theta'}\sin\theta\,dq + \left(\frac{q_{pp}}{q_p^2\theta'}\sin\theta - \cos\theta\right)q_p\,d\psi\right\}.$$

For a Bernoulli function which satisfies the partial differential equation

$$q_{pp} = Hq_p^2, \qquad H = H(q), \tag{31}$$

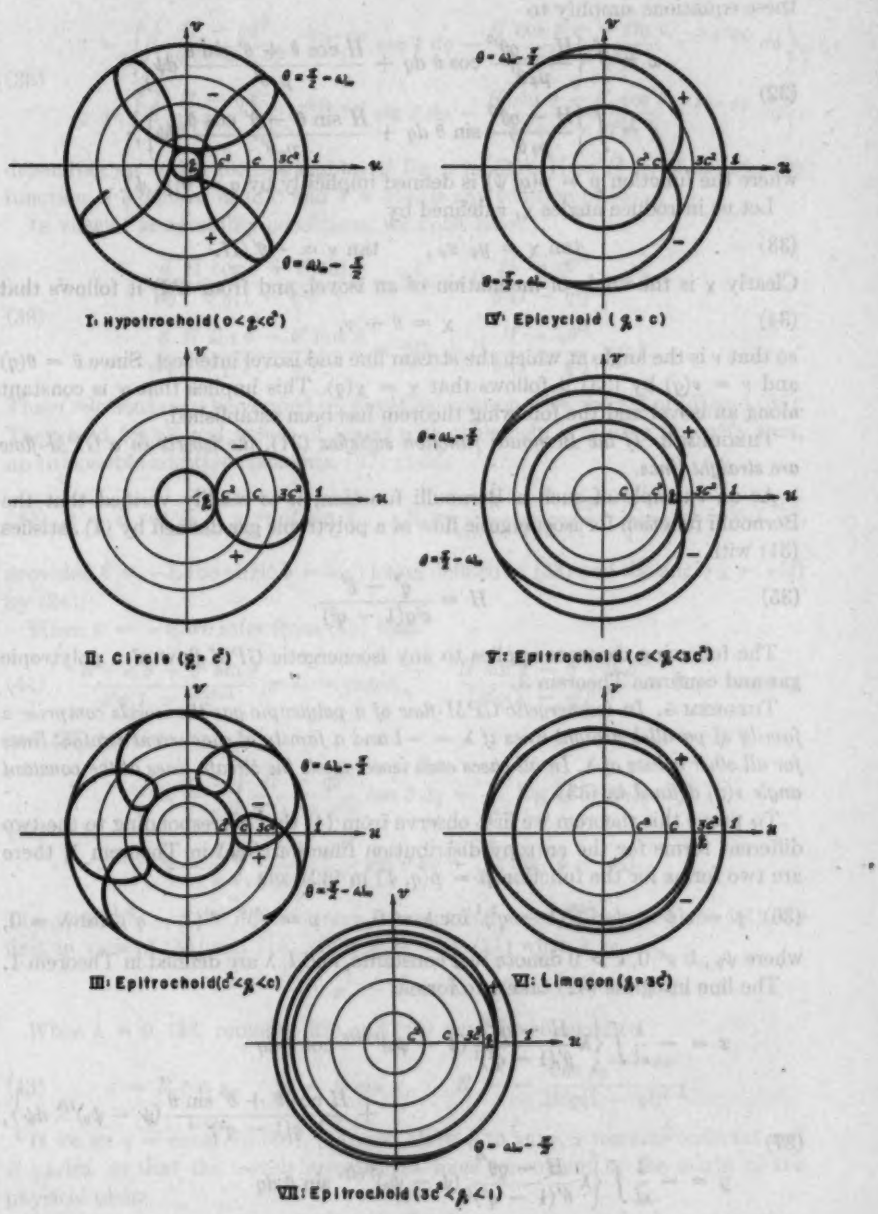
I: Hypotrochoid (0 < p < c²)
IV: Epicycloid (p = c)
II: Circle (p = c²)
V: Epitrochoid (c < p < 3c²)
III: Epitrochoid (c² < p < c)
VI: Limaçon (p = 3c²)
VII: Epitrochoid (3c² < p < 1)

FIG. 3

these equations simplify to

$$x = \int \left\{ \frac{H - q\theta'^2}{p_\psi \theta'} \cos\theta \, dq + \frac{H\cos\theta + \theta' \sin\theta}{p_q \theta'} \, d\psi \right\},$$
$$y = \int \left\{ \frac{H - q\theta'^2}{p_\psi \theta'} \sin\theta \, dq + \frac{H\sin\theta - \theta' \cos\theta}{p_q \theta'} \, d\psi \right\}, \tag{32}$$

where the function $p = p(q, \psi)$ is defined implicitly by $q = q(p, \psi)$.

Let us introduce angles χ, ν defined by

$$\tan\chi = y_\psi / x_\psi, \qquad \tan\nu = -\theta'/H. \tag{33}$$

Clearly χ is the angle of inclination of an isovel, and from (32) it follows that

$$\chi = \theta + \nu, \tag{34}$$

so that ν is the angle at which the stream line and isovel intersect. Since $\theta = \theta(q)$ and $\nu = \nu(q)$ by (33) it follows that $\chi = \chi(q)$. This implies that χ is constant along an isovel and the following theorem has been established.

THEOREM 3. *If the Bernoulli function satisfies* (31), *the isovels in a GPM-flow are straight lines.*

As an example of such a Bernoulli function, it is readily verified that the Bernoulli function for isoenergetic flow of a polytropic gas defined by (4) satisfies (31) with

$$H = \frac{q^2 - c^2}{c^2 q(1 - q^2)}. \tag{35}$$

The following theorem applies to any isoenergetic *GPM*-flow of a polytropic gas and confirms Theorem 3.

THEOREM 4. *In isoenergetic GPM-flow of a polytropic gas the isovels comprise a family of parallel straight lines if* $\lambda = -l$ *and a family of concurrent straight lines for all other values of* λ. *In all cases each isovel meets the stream lines at the constant angle* $\nu(q)$ *defined in* (33).

To prove this theorem we first observe from (4) that corresponding to the two different forms for the entropy distribution function $S(\psi)$ in Theorem 1, there are two forms for the function $p = p(q, \psi)$ in (32), viz.,

$$p = \kappa(\psi - \psi_0)^{-l/\lambda}(1 - q^2)^l \text{ for } \lambda \neq 0, \qquad p = e^{k(\psi - \psi_0)}(1 - q^2)^l \text{ for } \lambda = 0, \tag{36}$$

where ψ_0, $k \neq 0$, $\kappa > 0$ denote real constants, and l, λ are defined in Theorem 1.

The line integrals (32) take two forms

$$x = -\frac{1}{\kappa l} \int \left\{ \lambda \frac{H - q\theta'^2}{\theta'(1 - q^2)^l} (\psi - \psi_0)^{1 + l/\lambda} \cos\theta \, dq \right.$$
$$\left. + \frac{H\cos\theta + \theta' \sin\theta}{2\theta' q(1 - q^2)^{l-1}} (\psi - \psi_0)^{l/\lambda} \, d\psi \right\},$$
$$y = -\frac{1}{\kappa l} \int \left\{ \lambda \frac{H - q\theta'^2}{\theta'(1 - q^2)^l} (\psi - \psi_0)^{1 + l/\lambda} \sin\theta \, dq \right. \tag{37}$$
$$\left. + \frac{H\sin\theta - \theta' \cos\theta}{2\theta' q(1 - q^2)^{l-1}} (\psi - \psi_0)^{l/\lambda} \, d\psi \right\},$$

$$
\begin{aligned}
x &= \int \left\{ \frac{H - q\theta'^2}{k\theta'(1-q^2)^l} e^{-\lambda(\psi-\psi_0)} \cos\theta \, dq - \frac{H\cos\theta + \theta'\sin\theta}{2l\theta' q(1-q^2)^{l-1}} e^{-\lambda(\psi-\psi_0)} \, d\psi \right\}, \\
y &= \int \left\{ \frac{H - q\theta'^2}{k\theta'(1-q^2)^l} e^{-\lambda(\psi-\psi_0)} \sin\theta \, dq - \frac{H\sin\theta - \theta'\cos\theta}{2l\theta' q(1-q^2)^{l-1}} e^{-\lambda(\psi-\psi_0)} \, d\psi \right\},
\end{aligned} \tag{38}
$$

depending on which form is employed for p in (36). Here H is, of course, the function of q defined in (35) and $\theta = \theta(q)$ is any solution of (5).

In view of integrability conditions, we must have

$$
\begin{aligned}
\frac{d}{dq}\frac{H\cos\theta + \theta'\sin\theta}{\theta' q(1-q^2)^{l-1}} &= 2(\lambda + l)\frac{H - q\theta'^2}{\theta'(1-q^2)^l}\cos\theta, \\
\frac{d}{dq}\frac{H\sin\theta - \theta'\cos\theta}{\theta' q(1-q^2)^{l-1}} &= 2(\lambda + l)\frac{H - q\theta'^2}{\theta'(1-q^2)^l}\sin\theta.
\end{aligned} \tag{39}
$$

These relations may be verified by direct calculation with the aid of (5) and (35). They hold for all real values of λ and with their help it is easy to verify that, up to possible additive constants, (37) yields

$$
x = R\cos\chi, \quad y = R\sin\chi, \quad R = \frac{\lambda}{kl(\lambda + l)}\frac{\csc\nu}{2q(1-q^2)^{l-1}}(\psi - \psi_0)^{1+l/\lambda}, \tag{40}
$$

provided $\lambda \neq -l$, the angle $\nu = \nu(q)$ being defined in (33) and the angle $\chi = \chi(q)$ by (34).

When $\lambda = -l$, we infer from (39) that

$$
\frac{H\cos\theta + \theta'\sin\theta}{\theta' q(1-q^2)^{l-1}} = \alpha = \text{const.}, \qquad \frac{H\sin\theta - \theta'\cos\theta}{\theta' q(1-q^2)^{l-1}} = \beta = \text{const.}, \tag{41}
$$

and (37) leads to

$$
\begin{aligned}
x &= \frac{1}{\kappa}\int \frac{H - q\theta'^2}{\theta'(1-q^2)^l}\cos\theta \, dq - \frac{\alpha}{2\kappa l}\log(\psi - \psi_0), \\
y &= \frac{1}{\kappa}\int \frac{H - q\theta'^2}{\theta'(1-q^2)^l}\sin\theta \, dq - \frac{\beta}{2\kappa l}\log(\psi - \psi_0).
\end{aligned} \tag{42}
$$

The constants α, β are not arbitrary, for squaring and adding equations (41), we find, in view of (35) and (15), since $\Lambda = 1$ by (11) when $\lambda = -l$,

$$
\alpha^2 + \beta^2 = K > 0.
$$

When $\lambda = 0$, (38) replaces (37) and (40) must be changed to

$$
x = R\cos\chi, \quad y = R\sin\chi, \quad R = -\frac{\csc\nu e^{-\lambda(\psi-\psi_0)}}{2klq(1-q^2)^{l-1}}. \tag{43}
$$

If we set $q = \text{const.}$ in (40), (43) and allow ψ to vary, χ remains constant and R varies, so that the isovels are straight lines concurrent at the origin of the physical plane.

By placing $q = \text{const.}$ in (42) and allowing ψ to vary it is clear that when $\lambda = -l$, the isovels comprise a family of parallel straight lines of slope β/α.

This completes the proof of Theorem 4, but before passing on, it should be

noted that, whereas the isovels in irrotational PM-flows of a polytropic gas can envelope an arbitrary curve in the physical plane, the isovels in isoenergetic anisentropic GPM-flows are restricted to the special types in Theorem 4.

Returning for a moment to the case $\lambda = -l$, we have noted that (15) becomes

$$\theta'^2 = \frac{(q^2 - c^2)^2}{c^4 q^2 (1 - q^2)^2 [K q^2 (1 - q^2)^{2l-2} - 1]}. \tag{44}$$

For the quantity in the bracket to be positive in the interval $0 < q < 1$ it is necessary that

$$K > c^{-2}(1 - c^2)^{1-c^{-2}},$$

and if K meets this condition, this quantity will be positive in an interval

$$q_0 < q < q_1 \quad \text{where} \quad 0 < q_0 < c < q_1 < 1.$$

Consequently the hodograph curves lie in the annular ring bounded by the circles $q = q_0$, $q = q_1$. It is easy to obtain the equation of these hodograph curves in closed form. Indeed in (41) we have two homogeneous, simultaneous linear equations for θ', H and on equating the determinant of coefficients to zero one finds

$$\theta = \theta_0 + \operatorname{arc\,csc}\left(\sqrt{K} q (1 - q^2)^{l-1}\right), \qquad \theta_0 = \arctan \beta/\alpha,$$

from which it appears that the hodograph curves are simple closed curves tangent to the bounding circles $q = q_0$, $q = q_1$, with a radial line $\theta = \text{const.}$ as an axis of symmetry. The investigation of the flow in the physical plane is an interesting problem which we pass over in this paper.[6]

4. The flow in the physical plane for K = 0. From (32) we find

$$J = \partial(x, y)/\partial(q, \psi) = (q\theta'^2 - H)/p_q p_\phi \theta',$$

which, from (18), (35) yields

$$J = \pm \frac{q^2(q_0^2 - c^2)}{c^2 q_0 p_q p_\phi \sqrt{(1 - q^2)(q^2 - q_0^2)}}.$$

From (36) it follows that J remains finite and different from zero for all q in the interval $q_0 < q < 1$ and for all finite ψ if $\lambda = 0$. If $\lambda \neq 0$, we require instead $\psi > \psi_0$, unless $\lambda = -l$.

From (33), (35), (18′) we find

$$\tan \nu = \mp q_0 \sqrt{(1 - q^2)/(q^2 - q_0^2)}.$$

In particular $\tan \nu$ is negative if h is a $+$arc in Figure 3 and is positive for a $-$arc. If we introduce the angle ω in Figure 1 by substituting for q^2 from (30), we obtain

$$\tan \nu = -q_0 \cot c^2 \omega / q_0, \qquad -\pi \leqq \nu \leqq 0 \tag{45}$$

[6] See Prim, R. C., *Rotational gas flow around a corner*, Naval Ordnance Laboratory Memorandum 9642, September 1948. Prim employs a hodograph plane (u, v) in which u denotes the radial component and v the component of the velocity normal to the radius.

where

(46) $0 \leqq \omega \leqq \omega_m$ for a $+$arc, $\quad -\omega_m \leqq \omega \leqq 0$ for a $-$arc.

Let us use (45) to eliminate ν from (34) to obtain

(47) $$\theta = \chi + \arctan\,(q_0 \cot c^2\omega/q_0),$$

where we again choose the branch between 0 and π for the arctangent. For a $+$arc or a $-$arc we set $\theta_0 = 0$ in (30) and infer, for such arcs that

(48) $$\chi = \omega - \pi/2.$$

Consequently ω may be identified as the angle ω in the physical plane shown in Figure 4 between an isovel q = const. and the y-axis. This identification of ω,

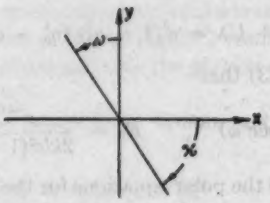

FIG. 4

which originally appeared as an angle in the hodograph plane in Figure 1, as an angle in the physical plane offers a geometrical construction of the velocity vector field in the physical plane analogous to the well-known construction for irrotational PM-flows. The construction is illustrated in Figure 5 where it is carried out for case I. ($0 < q_0 < c^2$).

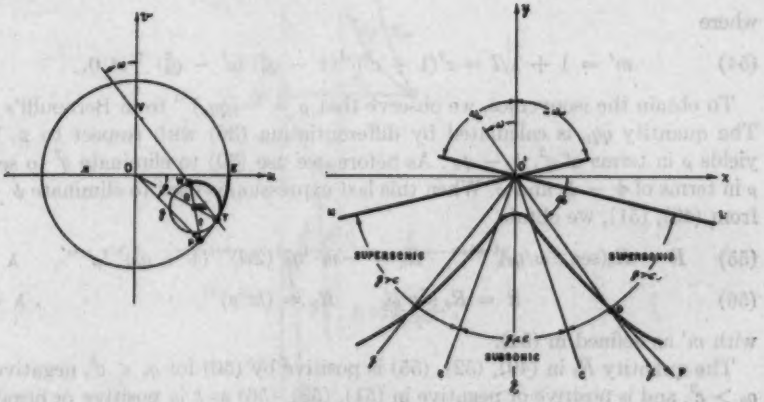

FIG. 5

FIG. 5

The angle ω being fixed in the hodograph plane we draw the ray $O'O$ in the physical plane parallel to the line VT in the hodograph plane. Thus $O'O$ makes an angle ω with y-axis and the angle χ with the x-axis by (48). From (40) the ray $O'O$ is an isovel at each point of which the velocity vector is the vector $\overrightarrow{OP}$ in the hodograph plane.

The form of the streamlines, isobars and isopycnics in the physical plane will now be investigated. We recall that for $K = 0$, only those values of λ outside the interval (20) need be considered.

When we substitute for q^2 from (30) and for ν from (45), we find that (40) yields

$$R = R_0\left(\sec\frac{c^2}{q_0}\omega\right)^{1/c^2}, \qquad R_0 = -\frac{(\psi-\psi_0)^m}{2\kappa lm\ q_0(1-q_0^2)^{l-1}}, \qquad \lambda \neq 0, \tag{49}$$

where

$$m = 1 + l/\lambda = c^2(1-q_0^2)\,(q_0^2-c^4)^{-1} \neq 0, \tag{50}$$

is finite, and from (19), (43) that

$$R = R_0(\sec\omega)^{1/c^2}, \qquad R_0 = \frac{e^{-k(\psi-\psi_0)}}{2klc^2(1-c^4)^{l-1}}, \qquad \text{for } \lambda = 0. \tag{51}$$

Equations (49), (51) are the polar equations for the stream lines in the physical plane. Before entering upon the discussion of their form we shall derive polar equations for the isobars and isopycnics.

For the isobars we use (30) to eliminate q^2 from (36), and then use the result to eliminate $\psi - \psi_0$ from (49), (51). One finds

$$R = R_0\left(\sec\frac{c^2}{q_0}\omega\right)^{-2\lambda-1}, \qquad R_0 = -\frac{\kappa^{\lambda/l}(1-q_0^2)^{\lambda+1}}{2lm\ q_0}\,p^{-m'}, \qquad \lambda \neq 0 \tag{52}$$

$$R = R_0\cos\omega, \qquad R_0 = (1-c^4)/2klc^2p \qquad \lambda = 0, \tag{53}$$

where

$$m' = 1 + \lambda/l = c^2(1+c^2)^{-1}(1-q_0^2)\,(c^2-q_0^2)^{-1} \neq 0. \tag{54}$$

To obtain the isopycnics, we observe that $\rho = -(qq_p)^{-1}$ from Bernoulli's law. The quantity qq_p is calculated by differentiating (36) with respect to p. This yields ρ in terms of q^2, $\psi - \psi_0$. As before, we use (30) to eliminate q^2 to secure ρ in terms of $\psi - \psi_0$ and ω. When this last expression is used to eliminate $\psi - \psi_0$ from (49), (51), we obtain

$$R = R_0(\sec c^2\omega/q_0)^{1-2\lambda/\gamma}, \quad R_0 = -m^{-1}q_0^{-1}(2\kappa l)^{\lambda/l}(1-q_0^2)^{\lambda/\gamma}\rho^{-m'}, \quad \lambda \neq 0 \tag{55}$$

$$R = R_0\sec\omega, \qquad R_0 = (kc^2\rho)^{-1}, \qquad \lambda = 0, \tag{56}$$

with m' as defined in (54).

The quantity R_0 in (49), (52), (55) is positive by (50) for $q_0 < c^2$, negative for $q_0 > c^2$, and is positive or negative in (51), (53), (56) as k is positive or negative. The isovel of minimum speed q_0, from (30), occurs when $\omega = 0$, and is, by (40), (48) the negative y-axis for $R_0 > 0$ but is the positive y-axis when $R_0 < 0$.

As q ranges from q_0 to 1 along a + arc in Figure 3, angle ω increases steadily from 0 to ω_m and the isovel turns counterclockwise from the isovel of minimum speed q_0 to the isovel of maximum speed 1. Under the same circumstances, for a − arc angle ω decreases steadily from 0 to $-\omega_m$ and the isovel turns clockwise from the isovel of minimum speed towards the isovel of maximum speed.

The values of ω_m taken for various values of q_0 are shown below.

q_0	0	c^2	$2c^2$	c	$3c^2$	$4c^2$	1
ω_m	0	$\pi/2$	π	$\pi/2c$	$3\pi/2$	2π	$\pi/2c^2$

For $\gamma = 7/5$, $c^2 = 1/6$ (the approximate values for air) the values of q_0 increase from left to right and $\omega_m \to 3\pi$ as $q_0 \to 1$. Since the field of flow in the physical plane covers a radial sector of angle $2\omega_m$ there will be no overlapping for $0 < q_0 < 2c^2$ and we obtain flows in the unshaded radial sectors shown in Figure 7. When $q_0 = 2c^2$ the isovels of maximum speed coincide on the negative y-axis and flow around the half line shown in III_4 of Figure 7 results. For $2c^2 < q_0 < 1$ the flow overlaps in the physical plane and only the portion corresponding to a − arc in Figure 3 is shown in Figure 8.

The polar equations of the streamlines, isobars and isopycnics given earlier all have the form

$$R = R_0(\sec c^2\omega/q_0)^e \tag{57}$$

with

$$e = e_\psi = 1/c^2, \qquad e = e_p = -(2\lambda + 1), \qquad e = e_\rho = 1 - 2\lambda/\gamma, \tag{58}$$

holding for streamlines, isobars and the isopycnics respectively. It is worth noting that (58) holds even for $\lambda = 0$, as a glance at (51), (53), (56) verifies. As an aid to graphing these curves these exponents are plotted against q_0 below.

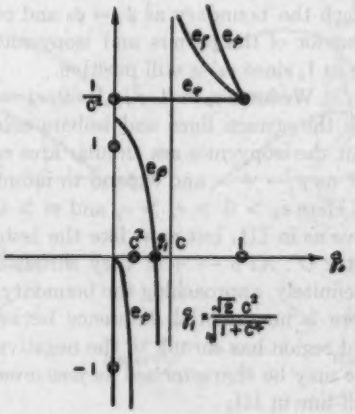

FIG. 6

From (18′) it is clear that $\theta' \gtrless 0$ for a + arc and $\theta' \lessgtr 0$ for a − arc according as $q \lessgtr c$. The stream lines accordingly have inflection points on the sonic isovels $q = c$ in I, II, III of Figures 7 and 8 and as indicated on these figures, their concavity differs in the subsonic and supersonic regions.

We now discuss the various flows in the physical plane shown in Figures 7 and 8. The roman numerals in these figures identify flows with their hodographs in Figure 3 and in the discussion which follows.

I. $0 < q_0 < c^2(-1/2 < \lambda < 0)$. From Figure 6 we see that $e_\psi > e_\rho > 0 > e_p$. In view of these inequalities the stream lines, isobars and isopycnics take the forms shown in I of Figure 7. An insight into the behavior of these curves for changing values of ψ, p, ρ is obtained by inspecting the various formulas for R_0 given in (49), (52), (55). In the present case $m < 0$, $m' > 0$, and we conclude that as $\psi \to +\infty$ the streamline approaches the boundary of the hatched region but for $\psi \to \psi_0$ it recedes to infinity below. As $p \to +\infty$ the isobars close down on the corner O' and as $p \to 0$ they expand, approaching the boundary. As $\rho \to +\infty$ the isopycnics approach the boundary and as $\rho \to 0$ they recede to infinity below, their behavior being not unlike that of the stream lines.

This case may be characterized as *corner flow*.

II. $q_0 = c^2(\lambda = 0)$. Here $e_\psi = 1/c^2, e_\rho = 1, e_p = -1$. The stream lines have the same general form as in I and, for $k > 0$, behave in the same way as $\psi \to +\infty$. The isobars are circles tangent to the boundary of the shaded region at O' and their diameters R_0, from (53), are inversely proportional to p. The isopycnics are horizontal straight lines whose distance R_0 below the boundary of the hatched region is inversely proportional to ρ. Thus when p and ρ are varied, the isobars and isopycnics exhibit the same general behavior as in I.

This case may be characterized as *half-plane flow*.

It is desirable to divide case III into the five subcases below.

III_1. $c^2 < q_0 < q_1\ (0 < \lambda < \gamma/2)$. Here $e_\psi > e_\rho > 0 > e_p$. Since $m > 0$, the stream lines approach the boundary as $\psi \to \psi_0$ and recede to infinity above as $\psi \to +\infty$. The behavior of the isobars and isopycnics as p, ρ are varied is essentially the same as in I, since m' is still positive.

III_2. $q_0 = q_1\ (\lambda = \gamma/2)$. We have $e_\psi = 1/c^2, e_\rho = 0, e_p = -(\gamma + 1)$ and $m > 0$, $m' > 0$. As ψ, p vary the stream lines and isobars exhibit the same general behavior as in III_1, but the isopycnics are circular arcs concentric at O'. These arcs shrink down on O' as $\rho \to +\infty$ and expand to infinity as $\rho \to 0$.

III_3. $q_1 < q_0 < 2c^2$. Here $e_\psi > 0 > e_\rho > e_p$ and $m > 0$, $m' > 0$. The stream lines and isobars behave as in III_2 but now, like the isobars, the isopycnics all emanate from the vertex O'. As $\rho \to +\infty$ they shrink down upon O' and as $\rho \to 0$ they expand indefinitely, approaching the boundary of the hatched region.

III_4. $q_0 = 2c^2$. There is no essential difference between this case and III_3 except that the hatched region has shrunk to the negative y-axis.

The four cases above may be characterized as *flow around a wedge*, the wedge degenerating into a half-line in III_4.

In the remaining cases $\omega_m > \pi$. As pointed out earlier, this leads to overlapping in the physical plane when both + and − arcs in the hodograph plane are

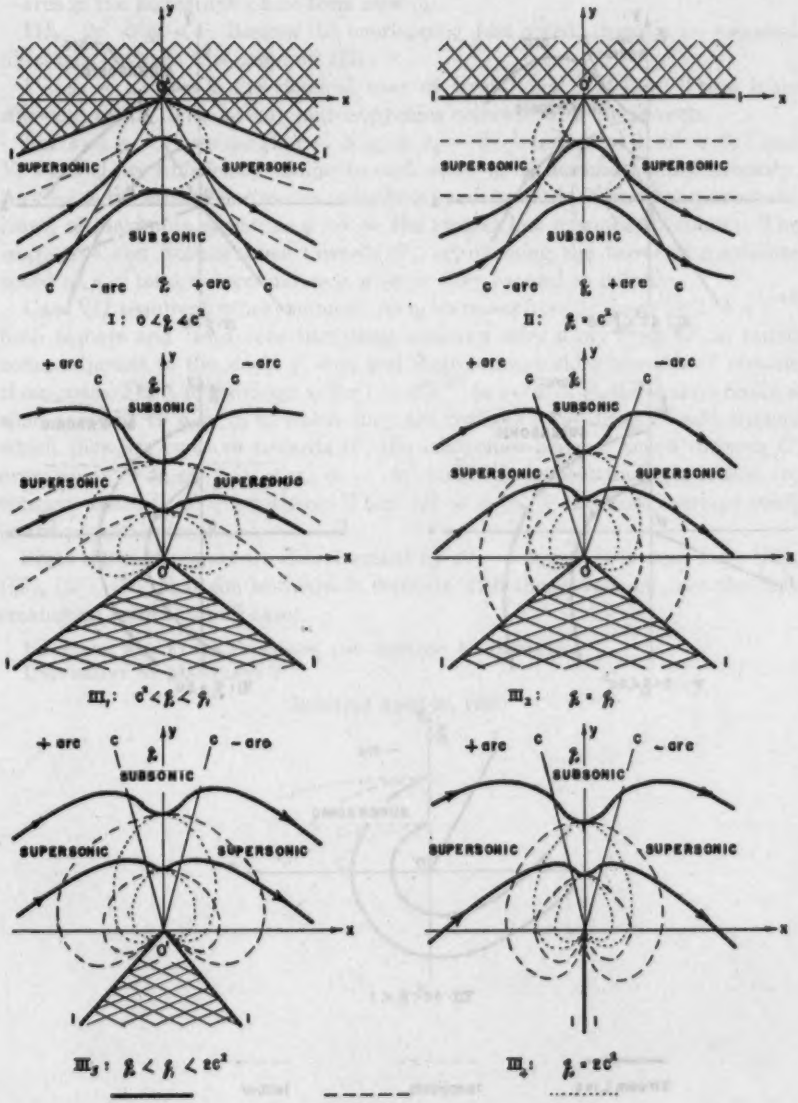
y
x
SUPERSONIC
SUPERSONIC
SUBSONIC
c
-arc
+arc
I: $0 < q_0 < c^2$
II: $q_0 = c^2$
III$_1$: $c^2 < q_0 < q_1$
III$_2$: $q_0 = q_1$
III$_3$: $q_0 < q_1 < 2c^2$
III$_4$: $q_0 = 2c^2$
Stream Line
Isopycnic
Isobar

FIG. 7

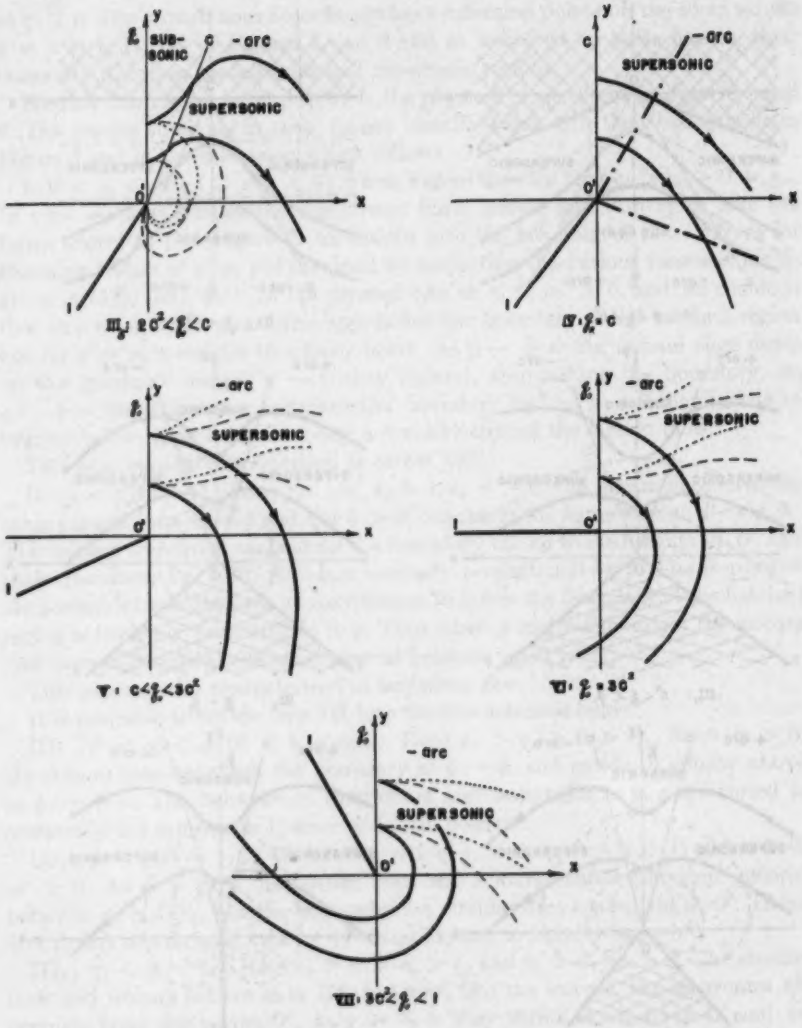
y
SUB-SONIC
C
-arc
SUPERSONIC
O'
x
1
III: 2C²<ρ/ρ₀<C
y
C
-arc
SUPERSONIC
O'
x
1
IV: ρ/ρ₀ = C
y
-arc
SUPERSONIC
O'
x
1
V: C<ρ/ρ₀<3C²
y
-arc
SUPERSONIC
O'
x
1
VI: ρ/ρ₀ = 3C²
y
1
-arc
SUPERSONIC
O'
x
VII: 3C²<ρ/ρ₀<1
Stream Line
Isopycnic
Isobar

FIG. 8

mapped upon the physical plane. For simplicity we confine our attention to $-$arcs in the hodograph plane from now on.

III_5. $2c^2 < q_0 < c$. Besides the overlapping just noted, there is no essential difference between this case and III_3.

IV. $q_0 = c$. This is the classical case of irrotational PM-flow. There is no subsonic region. The isobars and isopycnics coincide with the isovels.

For $c < q_0 < 1$ we see that $e_\rho > e_p > e_\psi = 1/c^2$ and $m > 0$, $m' < 0$. Cases V, VI, VII are sufficiently similar to each other to be discussed simultaneously. As $\psi \to \psi_0$ the stream line moves radially inwards towards O' and approaches the isovel of maximum speed; as $\psi \to \infty$ the stream line expands to infinity. The isopycnics and isobars move towards O', approaching the isovel of maximum speed as ρ, p tend to zero and as ρ, $p \to \infty$ they expand to infinity.

Case VII requires further comment. As q_0 increases from $3c^2$ to $\sqrt{2}c(1 + c^2)^{-1/2}$ both isobars and isopycnics turn their concave sides away from O' in radial zones adjacent to the isovel $q = q_0$ and their concave sides towards O' outside these zones. For q_0 in the range $\sqrt{2}c(1 + c^2)^{-1/2}$ to $c\sqrt{2 - c^2}$, the isobars retain a zone adjacent to $q = q_0$ in which they are concave away from O' and outside which they are concave towards O'; the isopycnics being concave towards O' everywhere. For $c\sqrt{2 - c^2} < q_0 < 4c^2$ both the isobars and isopycnics are concave towards O' everywhere. When $4c^2 < q_0 < 1$ the flow overlaps itself in the physical plane.

Since the isothermals are characterized by $p/\rho = \text{const.}$, it is clear from (52), (53), (55), (56) that the isothermals coincide with the isovels, i.e., are the rays emanating from O' in all cases.

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TABLES OF ROOTS AND INCOMPLETE INTEGRALS OF ASSOCIATED LEGENDRE FUNCTIONS OF FRACTIONAL ORDERS*

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General spherical harmonics and associated Legendre functions of non-integral orders are found to arise naturally in connection with an increasing number of problems in applied science. Qualitatively, such functions represent a well-known chapter of modern analysis; but quantitative investigations of their various properties are still mostly lacking. In recent months, in our laboratory the need arose for the roots of associated Legendre functions considered as functions of their index n, and for incomplete integrals of squares of such functions, in connection with problems related to the scattering of electromagnetic waves from conical-shaped conductors. Since no tables of these quantities have so far been published, they had to be evaluated ab initio and are reproduced in this paper with the hope that they may be found useful in future investigations involving associated Legendre functions of fractional orders. As the computation of the new tables gave rise to some special numerical problems, the first part of the present paper will be devoted to a description of the methods by which the tables were constructed; while its second part will contain the tables themselves.

Part I. Consider the equations

$$P_\mu^1(\cos\theta) = 0 \qquad (1), \qquad (d/d\theta)P_\nu^1(\cos\theta) = 0 \qquad (2)$$

where P_n^1 stands for associated Legendre functions of order 1 and fractional index μ or ν, and let it be required to ascertain such values of μ and ν for which these equations are satisfied for discrete values of $\pi/2 \leq \theta < \pi$.

Since, by definition,

$$P_\mu^1(\cos\theta) = \sin\theta\, d[P_\mu(\cos\theta)]/d\cos\theta, \tag{3}$$

equation 1 can be rewritten in terms of a hypergeometric series

$$P_\mu^1(\cos\theta) = \tfrac{1}{2}\mu(\mu+1)\sin\theta\, {}_2F_1\{\mu+2, 1-\mu; 2; \tfrac{1}{2}(1-\cos\theta)\} = 0 \tag{4}$$

which is satisfied, for any given θ, by an infinite spectrum of real values of μ. If the order k of the respective root is large, Pal[1] has proved that

$$\mu = \xi + \frac{\cot\theta}{\theta}\frac{1}{\xi}\left\{-0.375 + 0.1875\frac{1}{\xi} + 0.01171875\frac{1}{\xi^2} + 0.06152343\frac{\cot^2\theta}{\xi^2} + \cdots\right\}$$
$$+ \frac{\cot\theta}{\theta}\frac{1}{\xi^2}\left\{-0.140625 - 0.0703125\frac{1}{\xi^2} + \cdots\right\} + \cdots \tag{5}$$

where

$$\xi = (\pi/2\theta)(2k+\tfrac{1}{2}) - \tfrac{1}{2}. \tag{6}$$

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The foregoing asymptotic expansion converges provided that $\pi/2 \leq \theta \leq 3\pi/4$, and does so the more rapidly, the larger the value of k (i.e., the order of the respective root in the complete spectrum). For low values of k the formula obviously furnishes no useful result, so that the direct algebraic solution of equation (4) becomes indispensable. From the computational point of view, this becomes increasingly more difficult as $\theta_0 \to \pi$, as the convergence of the hypergeometric series on the left-hand side of (4) becomes then very slow and deteriorates further with increasing values of k. For this reason only the first few roots could be evaluated by this method, while higher roots were computed from (5) which converges the more rapidly, the greater the value of k. Equation (6) implies, in fact, that given any θ_0 and some ϵ arbitrarily small, positive, there exists some $k(\theta_0, \epsilon)$ such that, for all $k \geq k(\theta_0, \epsilon)$, the following holds true

$$| \mu_{k+1} - \mu_k - (\pi/\theta_0) | < \epsilon. \tag{7}$$

It must be noted, however, that for a given ϵ, $k(\theta_0, \epsilon)$ increases with θ_0. The larger the angle θ_0, the higher the rank of the root μ_k after which the recurrence relation (7) can be used with sufficient accuracy. These considerations determined the number of places with which the roots μ_k could be guaranteed for any k.

When we come to investigate the roots of equation (2) we find it convenient to put $\cos\theta = x$. For a given θ_0, the values of ν which satisfy equation (2) will also satisfy the equality

$$x(d/dx)P_\nu(x) = (1 - x^2)\,(d^2/dx^2)P_\nu(x). \tag{8}$$

Remembering that

$$P_\nu(x) = {}_2F_1(\nu + 1, -\nu, 1, (1 - x)/2)$$

and that

$$F(a, b, c - 1, z) - F(a, b, c, z) = [abz/c(c - 1)]\,F(a + 1, b + 1, c + 1, z),$$

equation (8) assumes the form

$$(1 + x)F(\nu + 2, 1 - \nu, 1, (1 - x)/2) = F(\nu + 2, 1 - \nu, 2, (1 - x)/2)$$

which is equivalent to

$$\sum_{j=0}^{\infty} \frac{(j + jx + x)\Gamma(\nu + j + 2)\Gamma(1 - \nu + j)}{j!\,\Gamma(\nu + 2)\Gamma(1 - \nu)\Gamma(j + 2)} \left(\frac{1 - x}{2}\right)^j = 0. \tag{9}$$

The foregoing equation admits also of an infinite number of discrete roots ν_k for any fixed value of x; and when k is large, these roots can again be approximated by a certain asymptotic expression deduced previously by Pal[1]. This expression is derived from Watson's asymptotic expansion for $P_n^m(\cos\theta)$[2] and remains valid for $\pi/6 < \theta < 5\pi/6$. A similar expression is then written for $P_{n-1}^m(\cos\theta)$; the substitution of $P_n^m(x)$ and $P_{n-1}^m(x)$ in the recursion formula

[1] Pal, B., Bull. of the Calcutta Math. Soc., 9, p. 85, 1918.

[2] Cf., Hobson, "Spherical and Ellipsoidal Harmonics", Cambridge 1931, p. 302.

$$nxP_n^m(x) - (n+m)P_{n-1}^m(x) = (x^2-1)\,(d/dx)P_n^m(x) = 0$$

then yields Pal's asymptotic expression for ν_k. To obtain the accuracy required by our problem, it was necessary to extend Pal's expressions to the terms of higher orders. The following discussion describes the procedure employed.

This equation was ordered in powers of $1/n$; terms of order $1/n^4$ and higher were neglected. The substitution

$$\psi = (\lambda - \lambda_0),$$

with

$$\lambda = n - \tfrac{1}{2}, \qquad \lambda_0 = (\pi/2\theta_0)(2k + \tfrac{1}{2}),$$

transforms Pal's equation into

$$\cos\left(\psi + \frac{\pi}{2} + k\pi\right)(2\sin\theta)^3 + \frac{n}{n+\frac{1}{2}}\cos\theta\sin(\psi + \theta + k\pi)(2\sin\theta)^3$$

$$- \frac{3}{2(2n+1)}\cos(\psi + \theta + k\pi)(2\sin\theta)^2$$

$$+ \frac{2n}{2n+1}\,\frac{(-3)}{2(2n+3)}\cos\theta\sin\left(\psi + 2\theta + \frac{3\pi}{2}\right)(2\sin\theta)^2 + \cdots$$

$$+ \frac{(-3)(5)}{2\cdot4\cdot(2n+1)(2n+3)}\cos\left(\psi + 2\theta - \frac{\pi}{2} + k\pi\right)(2\sin\theta)$$

$$+ \frac{(-3)(5)}{2\cdot4\cdot(2n+3)(2n+5)}\,\frac{2n}{(2n+1)}$$

$$\cdot\cos\theta\sin(\psi + 3\theta - \pi + k\pi)(2\sin\theta)$$

$$+ \frac{(-3)(5)(21)}{2\cdot4\cdot6\cdot(2n+1)(2n+3)(2n+5)}\cos(\psi + 3\theta - \pi + k\pi)$$

$$+ \frac{(-3)(5)(21)}{2\cdot4\cdot6\cdot(2n+3)(2n+5)(2n+7)}\,\frac{2n}{(2n+1)}$$

$$\cdot\cos\theta\sin\left(\psi + 4\theta - \frac{3\pi}{2} + k\pi\right) = 0. \quad (10)$$

Equation (10) is linear and homogeneous in $\sin\psi$ and $\cos\psi$ and can be written

$$\tan\psi = L/M. \quad (11)$$

We can then expand the right hand side of equation (11) as a power series in λ neglecting all terms of order $1/\lambda^4$ and higher. The coefficients of such a series are expressions in $\cos\theta_0$ and $\sin\theta_0$. We obtain

$$\tan(\lambda - \lambda_0)\theta_0 = \tan\psi = L/M = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + 0(1/\lambda^4),$$

[3] Cf., Whittaker and Watson, "Modern Analysis", Cambridge 1946, p. 324.
[4] Barnes, E. W., Quart. Journ. of Math., **39**, 97, 1908.

where

$$\alpha_0 = \frac{\alpha b_1}{A}, \qquad \alpha_1 = -\frac{\alpha_0}{A}\left\{a_2 + \alpha\frac{a_3}{\beta}\right\} + \frac{1}{A}\left\{b_2 + \frac{\alpha b_3}{\beta}\right\},$$

$$\alpha_2 = -\alpha_0\left\{\frac{a_4}{A\beta} + \frac{\alpha a_5}{A\beta\gamma} - \left(\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right)^2\right\} - \left\{\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right\}\left\{\frac{b_3}{A} + \frac{\alpha b_3}{A\beta}\right\} + \frac{b_4}{A\beta} + \frac{\alpha b_5}{A\beta\gamma},$$

$$\alpha_3 = -\alpha_0\left\{\frac{a_6}{A\beta\gamma} + \frac{\alpha a_7}{A\beta\gamma\delta} - 2\left(\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right)\left(\frac{a_4}{A\beta} + \frac{\alpha a_5}{A\beta\gamma}\right) + \left(\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right)^3\right\}$$
$$- \left\{\frac{a_4}{A\beta} + \frac{\alpha a_5}{A\beta\gamma} - \left(\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right)^2\right\}\left\{\frac{b_2}{A} + \frac{\alpha b_3}{A\beta}\right\}$$
$$- \left\{\frac{a_2}{A} + \frac{\alpha a_3}{A\beta}\right\}\left\{\frac{b_4}{A\beta} + \frac{\alpha b_5}{A\beta\gamma}\right\} + \frac{b_6}{A\beta\gamma} + \frac{\alpha b_7}{A\beta\gamma\delta},$$

and with the following definitions:

$$\alpha = 1 + 2\lambda, \qquad \beta = 2 + \lambda, \qquad \gamma = 3 + \lambda,$$
$$\delta = 4 + \lambda, \qquad A = 1 + a_1 + \lambda(1 + 2a_1)$$

$$a_1 = -\tfrac{1}{2}\cos\theta_0 \sin\left(\frac{\pi}{2} - \theta_0\right), \qquad a_2 = -\frac{3}{2^2}\frac{\cos(\pi/2 - \theta_0)}{2\sin\theta_0},$$

$$a_3 = \frac{3}{2^3}\cos\theta_0\frac{\sin(\pi - 2\theta_0)}{2\sin\theta_0}, \qquad a_4 = \frac{-3.5}{2^5}\frac{\cos(\pi - 2\theta_0)}{(2\sin\theta_0)^2},$$

$$a_5 = \frac{3.5}{2^6}\cos\theta_0\frac{\sin\left(\frac{3\pi}{2} - 3\theta_0\right)}{(2\sin\theta_0)^2}, \qquad a_6 = \frac{-3.5.7}{2^7}\frac{\cos\left(\frac{3\pi}{2} - 3\theta_0\right)}{(2\sin\theta_0)^3},$$

$$a_7 = \frac{3.5.7}{2^8}\cos\theta_0\frac{\sin(2\pi - 4\theta_0)}{(2\sin\theta_0)^3},$$

$$b_1 = \tfrac{1}{2}\cos\theta_0\cos\left(\frac{\pi}{2} - \theta_0\right), \qquad b_2 = \frac{-3}{2^2}\frac{\sin\left(\frac{\pi}{2} - \theta_0\right)}{2\sin\theta_0},$$

$$b_3 = \frac{-3}{2^3}\cos\theta_0\frac{\cos(\pi - 2\theta_0)}{2\sin\theta_0}, \qquad b_4 = \frac{-3.5}{2^5}\frac{\sin(\pi - 2\theta_0)}{(2\sin\theta_0)^2},$$

$$b_5 = \frac{-3.5}{2^6}\cos\theta_0\frac{\cos\left(\frac{3\pi}{2} - 3\theta_0\right)}{(2\sin\theta_0)^2}, \qquad b_6 = \frac{-3.5.7}{2^7}\frac{\sin\left(\frac{3\pi}{2} - 3\theta_0\right)}{(2\sin\theta_0)^3},$$

$$b_7 = \frac{-3.5.7}{2^8}\cos\theta_0\frac{\cos(2\pi - 4\theta_0)}{(2\sin\theta_0)^3}.$$

The roots ν_k are then defined by

$$\nu_k = \tfrac{1}{2} + \lambda_0 + (1/\theta_0)\tan^{-1}\omega \tag{12}$$

with $\omega = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \cdots$

Remembering that α_0, α_1, α_2 and α_3 are functions of λ and, therefore, of ν_k, we expand (12) by Lagrange's theorem

$$\nu_k = \tfrac{1}{2} + \lambda_0 + \frac{1}{\theta_0}(\tan^{-1}\omega)_{\lambda_0} + \frac{1}{\theta_0^2}\frac{d}{d\lambda}\left(\frac{1}{2!}\tan^{-1}\omega\right)^2_{\lambda_0} + \frac{1}{\theta_0^3}\frac{d^2}{d\lambda^2}\left(\frac{1}{3!}\tan^{-1}\omega\right)^3_{\lambda_0} + \frac{1}{\theta_0^4}\frac{d_4}{d\lambda^4}\left(\frac{1}{4!}\tan^{-1}\omega\right)^4_{\lambda_0} + \cdots. \quad (13)$$

This expression can be expanded in a power series of λ_0 whenever $\omega < 1$. The roots ν_k were thus obtained from equation (13) for all θ_0 and k in the ranges

$$\pi/2 \leq \theta_0 \leq 13\pi/18, \qquad k \leq 50.$$

If k increases beyond any limit, the corresponding roots can be approximated by the condition

$$\lim_{k\to\infty} | \nu_k - \nu_{k-1} - (\pi/\theta_0) | = 0.$$

If, however, k is in the neighborhood of unity, the convergence of equation (13) again becomes too slow to make it of any avail, so that recourse must be had to the original equation (9) considered as a function of ν. The same will be true—irrespective of k—when $\theta_0 \to \frac{3}{4}\pi$. If so, the value of $\alpha_{\max}$ as given by equation

$$\alpha_{0\max} = \frac{\cos\theta_0 \cos\left(\frac{\pi}{2} - \theta_0\right)}{1 - \cos\theta_0 \sin\left(\frac{\pi}{2} - \theta_0\right)}$$

approaches -1 and the error committed in neglecting higher powers of α_0 in (13) becomes intolerable. It is this reason which prevented us from extending our tables of the ν_k's which follow beyond $\theta_0 = 130°$.

Part II. Having specified the roots of the equation (1) and (2), let us now set out to evaluate the definite integrals of the form

$$B_n(x_0) = \int_{x_0}^{1} [P^1_n(x)]^2\, dx, \quad (14)$$

where $n \equiv \mu$ or ν are the roots of equation (1) or (2) for $x = x_0$. If $x_0 = -1$ the answer is well known[3]; but for $-1 < x_0 < 0$ the task proves to be by no means as easy.

In order to accomplish it, let us consider the integral (14) as

$$B_n = I_1 + I_2,$$

where

$$I_1 = \int_{x_0}^{0} [P^1_n(x)]^2\, dx \quad \text{and} \quad I_2 = \int_{0}^{1} [P^1_n(x)]^2\, dx.$$

As long as x_0 is small, I_1 can be evaluated if we approximate its integrand by Pal's asymptotic formula

$$P_n^1(\cos\theta) \approx (2n/\pi \sin\theta)^{\frac{1}{2}}\{u\cos\lambda + v\sin\lambda\},$$

where

$$\lambda = (n+\tfrac{1}{2})\theta + \pi/4$$

and

$$u = 1 + \frac{1}{4n} + \frac{3c_2}{4n^2} + \cdots \qquad v = -\frac{3}{8n}\cot\theta - \frac{3c_2'}{4n^2} + \cdots$$

$$c_2 = \tfrac{1}{24} + \tfrac{5}{24}\cot^2\theta \qquad c_2' = -\tfrac{1}{8}\cot\theta.$$

These coefficients are complete only as far as terms of the order of n^{-2} are concerned. Inasmuch as terms of higher orders in n^{-1} become exceedingly complicated, it is evident that Pal's asymptotic formula will be useful only for large values of n and if θ does not depart too much from 90°. Within the scheme of our approximations we can, therefore, write

$$\{P_n^1(\cos\theta)\}^2 \approx (2n/\pi\sin\theta)\{u^2\cos^2\lambda + 2uv\sin\lambda\cos\lambda + v^2\sin^2\lambda\} \quad (15)$$

where, to the order of accuracy within which we are working,

$$u^2 = \alpha + \beta\cot^2\theta + \cdots \quad v^2 = \gamma\cot^2\theta + \cdots \quad 2uv = \delta\cot\theta + \cdots$$

$$\alpha = 1 + \frac{1}{2n} + \frac{1}{8n^2} + \cdots \quad \beta = \frac{15}{64n^2} + \cdots \quad \gamma = \frac{9}{64n^2} + \cdots$$

$$\delta = -\frac{3}{4n} + \cdots \quad (16)$$

Inserting (16) in (15) and integrating we obtain

$$I_1 = \frac{n}{\pi}\left\{\alpha\left[\theta_0 - \frac{\pi}{2} + \frac{\cos p\theta_0}{p} - \frac{\cos(\pi p/2)}{p}\right]\right.$$

$$\left. - (\beta+\gamma)\left[\cot\theta_0 + \theta_0 - \frac{\pi}{2}\right] - (\beta-\gamma)F_1(p) + \delta F_2(p)\right\},$$

where

$$F_1(p) = \int_{\pi/2}^{\theta_0}\cot^2\theta\sin p\theta\,d\theta, \qquad F_2(p) = \int_{\pi/2}^{\theta_0}\cot\theta\cos p\theta\,d\theta$$

and $p = 2n+1$. The latter two expressions cannot be integrated in a simple expression for all p but for any one p the evaluation by quadratures presents no difficulties.

When we come to integrate the expression for I_2, we find it convenient to consider it, by virtue of equation (3), in the form

$$I_2 = \int_0^1 (1-x^2)\{(d/dx)P_n(x)\}^2\,dx.$$

If we integrate by parts and remember that $P_n(x)$ satisfies the Legendre equation, the foregoing expression clearly reduces to

$$I_2 = -P_n(0)(d/dx)P_n(0) + n(n+1)\int_0^1 \{P_n(x)\}^2\,dx.$$

From the relations

$$P_n(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)} \quad \text{and} \quad \frac{d}{dx}P_n(0) = \frac{-2\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)\Gamma\left(-\frac{n}{2}\right)}$$

we find that

$$P_n(0)\cdot(d/dx)P_n(0) = \sin n\pi/\pi.$$

Now Barnes[4] proved that, for unrestricted n,

$$\int_0^1 \{P_n(x)\}^2\,dx = {}_3F_2\left\{\begin{matrix} n+1, & -n, & \frac{1}{2} \\ 1 & , & \frac{3}{2} \end{matrix}\;;\; 1\right\}; \tag{17}$$

but the generalized hypergeometric series on the right-hand side converges, unfortunately, so slowly that several thousand terms would be required to stabilize the first significant figure of the sum. The series as it stands is, therefore, quite useless for approximating the numerical value of the integral on the left-hand side; but the difficulty can be overcome by the following device.

Consider the function

$$y_n(z) = {}_3F_2\left\{\begin{matrix} n+1, & -n, & \frac{1}{2} \\ 1 & , & \frac{3}{2} \end{matrix}\;;\; z\right\} \tag{18}$$

which is known[5] to satisfy the third-order differential equation

$$z^2(1-z)y''' + \tfrac{1}{2}z(7-9z)y'' + \{\tfrac{1}{2} - 3z + n(n+1)z\}y' + \tfrac{1}{2}n(n+1)y = 0.$$

Suppose that this equation is integrated numerically from the initial conditions implied in (18) for $z = 0$ up to $z = 1$; the value of $y_n(1)$ is clearly equal to the sum of the series on the right-hand side of equation (17) and, therefore, to the desired integral on its left-hand side. We have performed a sufficient number of such integrations to enable us to prepare a table of $y_n(1)$ for $n = 0.00(0.05)1.00$; if $n > 1$, the appropriate value of $y_n(1)$ can be found from a repeated use of the recursion formula

$$(2n+1)y_n(1) = (2n-1)y_{n-1}(1) + 2\sin n\pi/n\pi.$$

In conclusion, it is a pleasure to record that the construction of the present tables was undertaken at the suggestion of Dr. Roy C. Spencer and of Mr. C. Sletten of the Cambridge Field Station of Watson Laboratories. The authors wish to express their appreciation to Professor Zdeněk Kopal for his guidance

[4] Bailey, W. N., "Generalized Hypergeometric Series", Cambridge Tracts on Mathematics and Mathematical Physics, No. 32, Cambridge, 1935.

TABLE I

First fifty roots μ_k of the equation $P^1_{\mu_k}(\cos\theta_0) = 0$ for $\theta_0 = 90°(5)175°$. For $\theta_0 \leq 130°$ the first few roots were improved by means of the appropriate hypergeometric series

k	$\theta_0 = 90°$	$\theta_0 = 95°$	$\theta_0 = 100°$	$\theta_0 = 105°$	$\theta_0 = 110°$
1	2.000	1.8763	1.7659	1.6669	1.5775
2	4.000	3.7677	3.5591	3.3708	3.2005
3	6.000	5.6611	5.3564	5.0810	4.8310
4	8.000	7.5551	7.1549	6.7931	6.4647
5	10.000	9.4493	8.9540	8.5060	8.0991
6	12.000	11.344	10.753	10.219	9.7342
7	14.000	13.238	12.553	11.933	11.370
8	16.000	15.133	14.353	13.647	13.005
9	18.000	17.027	16.152	15.361	14.641
10	20.000	18.922	17.952	17.075	16.277
11	22.000	20.817	19.752	18.789	17.913
12	24.000	22.711	21.552	20.503	19.549
13	26.000	24.606	23.352	22.217	21.185
14	28.000	26.501	25.151	23.931	22.821
15	30.000	28.395	26.951	25.645	24.457
16	32.000	30.290	28.751	27.359	26.094
17	34.000	32.185	30.551	29.073	27.730
18	36.000	34.080	32.351	30.787	29.366
19	38.000	35.974	34.151	32.502	31.002
20	40.000	37.869	35.951	34.216	32.639
21	42.000	39.764	37.751	35.930	34.275
22	44.000	41.658	39.551	37.644	35.911
23	46.000	43.553	41.351	39.359	37.547
24	48.000	45.448	43.151	41.073	39.184
25	50.000	47.343	44.951	42.787	40.820
26	52.000	49.237	46.751	44.501	42.456
27	54.000	51.132	48.551	46.215	44.093
28	56.000	53.027	50.351	47.930	45.729
29	58.000	54.921	52.151	49.644	47.365
30	60.000	56.816	53.951	51.358	49.001
31	62.000	58.711	55.751	53.072	50.638
32	64.000	60.606	57.551	54.787	52.274
33	66.000	62.500	59.351	56.501	53.910
34	68.000	64.395	61.151	58.215	55.547
35	70.000	66.290	62.951	59.929	57.183
36	72.000	68.184	64.751	61.644	58.819
37	74.000	70.079	66.551	63.358	60.456
38	76.000	71.974	68.351	65.072	62.092
39	78.000	73.869	70.151	66.787	63.728
40	80.000	75.763	71.951	68.501	65.365
41	82.000	77.658	73.751	70.215	67.001
42	84.000	79.553	75.550	71.929	68.637
43	86.000	81.448	77.350	73.644	70.274
44	88.000	83.342	79.150	75.358	71.910
45	90.000	85.237	80.950	77.072	73.546
46	92.000	87.132	82.750	78.786	75.183
47	94.000	89.027	84.550	80.501	76.819
48	96.000	90.921	86.350	82.215	78.455
49	88.000	92.816	88.150	83.929	80.092
50	100.000	94.711	89.950	85.643	81.728

TABLE I—*Continued*

k	$\theta_0 = 115°$	$\theta_0 = 120°$	$\theta_0 = 125°$	$\theta_0 = 130°$	$\theta_0 = 135°$
1	1.4969	1.4241	1.3580	1.3001	1.249
2	3.0456	2.9043	2.7753	2.6569	2.549
3	4.6037	4.3956	4.2050	4.0297	3.869
4	6.1651	5.8910	5.6394	5.4077	5.194
5	7.7279	7.3880	7.0758	6.7881	6.522
6	9.2915	8.8860	8.5133	8.1697	7.852
7	10.855	10.384	9.9515	9.5522	9.183
8	12.420	11.883	11.390	10.935	10.514
9	13.984	13.382	12.829	12.318	11.846
10	15.549	14.882	14.268	13.702	13.178
11	17.114	16.381	15.707	15.086	14.511
12	18.679	17.881	17.147	16.470	15.843
13	20.243	19.380	18.586	17.854	17.176
14	21.808	20.880	20.026	19.238	18.508
15	23.373	22.380	21.465	20.622	19.841
16	24.938	23.879	22.905	22.006	21.174
17	26.503	25.379	24.345	23.390	22.507
18	28.068	26.879	25.785	24.775	23.840
19	29.633	28.379	27.224	26.159	25.173
20	31.198	29.878	28.664	27.543	26.506
21	32.763	31.378	30.104	28.928	27.839
22	34.329	32.878	31.544	30.312	29.172
23	35.894	34.378	32.984	31.697	30.505
24	37.459	35.878	34.423	33.081	31.838
25	39.024	37.378	35.863	34.465	33.171
26	40.589	38.878	37.303	35.850	34.505
27	42.154	40.378	38.743	37.234	35.838
28	43.719	41.877	40.183	38.619	37.171
29	45.285	43.377	41.623	40.003	38.504
30	46.850	44.877	43.063	41.388	39.837
31	48.415	46.377	44.503	42.772	41.170
32	49.980	47.877	45.943	44.157	42.504
33	51.545	49.377	47.383	45.541	43.837
34	53.110	50.877	48.822	46.926	45.170
35	54.675	52.377	50.262	48.310	46.503
36	56.241	53.877	51.702	49.695	47.837
37	57.806	55.377	53.142	51.080	49.170
38	59.371	56.877	54.582	52.464	50.503
39	60.936	58.377	56.022	53.849	51.836
40	62.501	59.877	57.462	55.233	53.170
41	64.067	61.377	58.902	56.618	54.503
42	65.632	62.877	60.342	58.002	55.836
43	67.197	64.377	61.782	59.387	57.169
44	68.762	65.877	63.222	60.771	58.503
45	70.327	67.377	64.662	62.156	59.836
46	71.893	68.876	66.102	63.541	61.169
47	73.458	70.376	67.542	64.925	62.503
48	75.023	71.876	68.982	66.310	63.836
49	76.588	73.376	70.422	67.694	65.169
50	78.153	74.876	71.862	69.079	66.502

TABLE I—*Continued*

k	$\theta_0 = 140°$	$\theta_0 = 145°$	$\theta_0 = 150°$	$\theta_0 = 155°$	$\theta_0 = 160°$
1	1.202	1.159	1.122	1.092	1.069
2	2.451	2.360	2.278	2.204	2.138
3	3.721	3.585	3.460	3.344	3.240
4	4.997	4.815	4.646	4.491	4.351
5	6.277	6.049	5.838	5.643	5.465
6	7.558	7.285	7.032	6.798	6.582
7	8.841	8.523	8.228	7.954	7.700
8	10.124	9.762	9.425	9.111	8.820
9	11.408	11.001	10.622	10.269	9.941
10	12.692	12.241	11.820	11.428	11.063
11	13.977	13.481	13.018	12.587	12.185
12	15.262	14.721	14.217	13.746	13.308
13	16.546	15.961	15.416	14.906	14.431
14	17.831	17.202	16.614	16.066	15.554
15	19.116	18.442	17.814	17.226	16.678
16	20.402	19.683	19.013	18.387	17.801
17	21.687	20.924	20.212	19.547	18.925
18	22.972	22.165	21.411	20.708	20.049
19	24.257	23.405	22.611	21.868	21.173
20	25.543	24.646	23.810	23.029	22.297
21	26.828	25.887	25.010	24.189	23.421
22	28.114	27.128	26.209	25.350	24.546
23	29.399	28.369	27.409	26.511	25.670
24	30.684	29.610	28.609	27.672	26.795
25	31.970	30.852	29.808	28.833	27.919
26	33.255	32.093	31.008	29.994	29.044
27	34.541	33.334	32.208	31.155	30.168
28	35.826	34.575	33.407	32.316	31.293
29	37.112	35.816	34.607	33.476	32.417
30	38.398	37.057	35.807	34.637	33.542
31	39.683	38.299	37.007	35.798	34.667
32	40.969	39.540	38.206	36.960	35.791
33	42.254	40.781	39.406	38.121	36.916
34	43.540	42.022	40.606	39.282	38.041
35	44.825	43.263	41.806	40.443	39.166
36	46.111	44.505	43.006	41.604	40.290
37	47.397	45.746	44.206	42.765	41.415
38	48.682	46.987	45.405	43.926	42.540
39	49.968	48.228	46.605	45.087	43.665
40	51.254	49.470	47.805	46.248	44.789
41	52.539	50.711	49.005	47.409	45.914
42	53.825	51.952	50.205	48.571	47.039
43	55.110	53.194	51.405	49.732	48.164
44	56.396	54.435	52.605	50.893	49.289
45	57.682	55.676	53.805	52.054	50.414
46	58.967	56.917	55.004	53.215	51.538
47	60.253	58.159	56.204	54.376	52.663
48	61.539	59.400	57.404	55.538	53.788
49	62.824	60.641	58.604	56.699	54.913
50	64.110	61.883	59.804	57.860	56.038

TABLE I—*Concluded*

k	$\theta_0 = 165°$	$\theta_0 = 170°$	$\theta_0 = 175°$
1	1.053	1.05	1.00
2	2.083	2.04	2.01
3	3.150	3.07	3.03
4	4.225	4.113	4.04
5	5.304	5.159	5.08
6	6.385	6.208	6.08
7	7.467	7.259	7.08
8	8.552	8.309	8.098
9	9.637	9.362	9.120
10	10.724	10.415	10.145
11	11.811	11.469	11.169
12	12.899	12.524	12.193
13	13.988	13.579	13.217
14	15.076	14.634	14.241
15	16.165	15.690	15.266
16	17.254	16.747	16.291
17	18.344	17.803	17.316
18	19.433	18.860	18.341
19	20.523	19.917	19.366
20	21.613	20.974	20.392
21	22.703	22.031	21.418
22	23.793	23.089	22.444
23	24.883	24.146	23.470
24	25.973	25.204	24.497
25	27.063	26.262	25.524
26	28.153	27.320	26.550
27	29.244	28.378	27.577
28	30.334	29.436	28.604
29	31.424	30.494	29.631
30	32.515	31.552	30.658
31	33.605	32.610	31.685
32	34.696	33.668	32.713
33	35.786	34.726	33.740
34	36.877	35.784	34.768
35	37.967	36.843	35.795
36	39.058	37.901	36.823
37	40.148	38.959	37.850
38	41.239	40.018	38.878
39	42.330	41.076	39.906
40	43.420	42.134	40.933
41	44.511	43.193	41.962
42	45.601	44.251	42.989
43	46.692	45.310	44.017
44	47.783	46.368	45.045
45	48.873	47.427	46.073
46	49.964	48.485	47.101
47	51.055	49.544	48.129
48	52.146	50.602	49.156
49	53.236	51.661	50.185
50	54.327	52.719	51.213

TABLE II

First fifty roots ν_k of the equation: $\{d/d\theta P^1_k(\cos\theta)\}_{\theta=\theta_0} = 0$ for $\theta_0 = 90°(5)130°$. It is to be noted that a Lagrangian five-point interpolation formula allows the interpolation θ—wise of the roots μ_k and ν_k corresponding to any given k

k	$\theta_0 = 90°$	$\theta_0 = 95°$	$\theta_0 = 100°$	$\theta_0 = 105°$	$\theta_0 = 110$
1	1.000	.9572	.9216	.8921	.8696
2	3.000	2.8301	2.6788	2.5437	2.4230
3	5.000	4.7195	4.4681	4.2417	4.0373
4	7.000	6.6116	6.2632	5.9486	5.6636
5	9.000	8.5052	8.0604	7.6586	7.2942
6	11.000	10.399	9.8586	9.3702	8.9268
7	13.000	12.293	11.657	11.082	10.560
8	15.000	14.187	13.456	12.795	12.195
9	17.000	16.082	15.256	14.508	13.830
10	19.000	17.976	17.055	16.222	15.465
11	21.000	19.871	18.855	17.936	17.100
12	23.000	21.765	20.654	19.649	18.736
13	25.000	23.660	22.454	21.363	20.372
14	27.000	25.554	24.254	23.077	22.007
15	29.000	27.449	26.053	24.791	23.643
16	31.000	29.344	27.853	26.505	25.279
17	33.000	31.238	29.653	28.219	26.915
18	35.000	33.133	31.453	29.933	28.551
19	37.000	35.028	33.253	31.647	30.187
20	39.000	36.922	35.052	33.361	31.823
21	41.000	38.817	36.852	35.075	33.459
22	43.000	40.712	38.652	36.789	35.095
23	45.000	42.606	40.452	38.503	36.731
24	47.000	44.501	42.252	40.217	38.367
25	49.000	46.396	44.052	41.932	40.003
26	51.000	48.290	45.852	43.646	41.639
27	53.000	50.185	47.652	45.360	43.275
28	55.000	52.080	49.452	47.074	44.911
29	57.000	53.974	51.252	48.788	46.547
30	59.000	55.869	53.052	50.502	48.183
31	61.000	57.764	54.852	52.217	49.819
32	63.000	59.659	56.652	53.931	51.455
33	65.000	61.553	58.452	55.645	53.091
34	67.000	63.448	60.251	57.359	54.727
35	69.000	65.343	62.051	59.074	56.363
36	71.000	67.238	63.851	60.788	57.999
37	73.000	69.132	65.651	62.502	59.635
38	75.000	71.027	67.451	64.216	61.271
39	77.000	72.922	69.251	65.930	62.907
40	79.000	74.816	71.051	67.645	64.543
41	81.000	76.711	72.851	69.359	66.179
42	83.000	78.606	74.651	71.073	67.815
43	85.000	80.501	76.451	72.787	69.451
44	87.000	82.395	78.251	74.502	71.087
45	89.000	84.290	80.051	76.216	72.723
46	91.000	86.185	81.851	77.930	74.359
47	93.000	88.080	83.651	79.644	75.995
48	95.000	89.974	85.451	81.359	77.631
49	97.000	91.869	87.251	83.073	79.267
50	99.000	93.764	89.051	84.787	80.903
51	101.000	95.659	90.851	86.501	82.539

TABLE II—*Concluded*

k	$\theta_0 = 115°$	$\theta_0 = 120°$	$\theta_0 = 125°$	$\theta_0 = 130°$
1	.8532	.8434	.8403	.8458
2	2.3151	2.2191	2.1352	2.0611
3	3.8522	3.6843	3.5321	3.3946
4	5.4045	5.1683	4.9527	4.7560
5	6.9623	6.6591	6.3816	6.1275
6	8.5227	8.1532	7.8145	7.5038
7	10.084	9.6490	9.2495	8.8827
8	11.647	11.146	10.686	10.263
9	13.210	12.644	12.123	11.645
10	14.774	14.142	13.561	13.027
11	16.338	15.640	14.999	14.409
12	17.902	17.139	16.437	15.792
13	19.467	18.638	17.876	17.175
14	21.031	20.137	19.315	18.559
15	22.596	21.636	20.754	19.942
16	24.160	23.135	22.193	21.326
17	25.725	24.635	23.633	22.710
18	27.290	26.134	25.072	24.094
19	28.855	27.634	26.511	25.478
20	30.420	29.133	27.951	26.862
21	31.986	30.633	29.390	28.246
22	33.551	32.133	30.830	29.630
23	35.116	33.633	32.270	31.015
24	36.681	35.133	33.710	32.399
25	38.246	36.633	35.150	33.784
26	39.812	38.133	36.590	35.169
27	41.377	39.633	38.030	36.553
28	42.942	41.133	39.470	37.938
29	44.507	42.633	40.910	39.323
30	46.072	44.133	42.350	40.707
31	47.638	45.633	43.790	42.092
32	49.203	47.133	45.230	43.476
33	50.768	48.633	46.670	44.861
34	52.333	50.133	48.110	46.246
35	53.899	51.633	49.550	47.630
36	55.464	53.133	50.990	49.015
37	57.029	54.633	52.430	50.399
38	58.594	56.133	53.870	51.784
39	60.160	57.633	55.310	53.169
40	61.725	59.133	56.750	54.553
41	63.290	60.633	58.190	55.938
42	64.855	62.133	59.630	57.322
43	66.420	63.633	61.070	58.707
44	67.986	65.133	62.510	60.092
45	69.551	66.633	63.950	61.476
46	71.116	68.133	65.390	62.861
47	72.681	69.633	66.830	64.245
48	74.247	71.133	68.270	65.630
49	75.812	72.633	69.710	67.015
50	77.377	74.133	71.150	68.399
51	78.942	75.633	72.590	69.784

TABLE III

Values of the integral $I_1 = \int_{x_0}^{0} \{P_\mu^1(x)\}^2\,dx$ *where* $x_0 = \cos\theta_0$, *for all* μ_k*'s obtained from Table I for* $\theta_0 = 95°(5)145°$

k	$\theta_0 = 95°$	$\theta_0 = 100°$	$\theta_0 = 105°$	$\theta_0 = 110°$	$\theta_0 = 115°$
1	.00167	.0115	.0329	.0656	.107
2	.01032	.0664	.174	.3174	.474
3	.0306	.1814	.427	.6771	.866
4	.0645	.3512	.707	.9353	1.009
5	.117	.5489	.912	1.006	1.025
6	.188	.7380	.9960	1.020	1.236
7	.274	.8838	1.0037	1.177	1.650
8	.3697	.9676	1.0365	1.498	1.951
9	.4738	.9990	1.1774	1.826	2.005
10	.5784	1.0016	1.4278	1.985	2.073
11	.6790	1.013	1.7058	2.002	2.391
12	.7705	1.068	1.9110	2.070	2.800
13	.8481	1.183	1.9937	2.321	2.992
14	.9093	1.353	2.0019	2.675	3.009
15	.9536	1.550	2.0362	2.930	3.170
16	.9811	1.738	2.1780	2.999	3.565
17	.9952	1.883	2.4288	3.016	3.911
18	1.000	1.968	2.7064	3.172	4.003
19	1.000	1.998	2.9096	3.497	4.041
20	1.002	2.001	2.9934	3.824	4.311
21	1.011	2.013	3.0015	3.983	4.729
22	1.031	2.068	3.0359	4.000	4.975
23	1.067	2.176	3.1777	4.068	5.003
24	1.119	2.352	3.4283	4.320	5.114
25	1.188	2.549	3.7061	4.674	5.478
26	1.274	2.737	3.9098	4.927	5.861
27	1.370	2.883	3.9924	4.997	5.999
28	1.473	2.968	4.0012	5.014	6.019
29	1.579	2.998	4.0355	5.170	6.235
30	1.679	3.001	4.1782	5.496	6.649
31	1.770	3.013	4.4295	5.823	6.948
32	1.848	3.068	4.7061	5.981	7.002
33	1.909	3.183	4.9095	5.998	7.071
34	1.953	3.352	4.9925	6.066	7.392
35	1.981	3.550	5.0010	6.319	7.800
36	1.995	3.737	5.0358	6.673	7.990
37	2.000	3.882	5.1782	6.926	8.007
38	2.000	3.968	5.4289	6.996	8.169
39	2.002	3.998	5.7055	7.012	8.565
40	2.011	4.001	5.9095	7.169	8.910
41	2.031	4.013	5.9925	7.495	9.001
42	2.067	4.069	6.0005	7.822	9.039
43	2.119	4.183	6.0352	7.979	9.310
44	2.189	4.354	6.1781	7.996	9.728
45	2.274	4.550	6.4291	8.065	9.974
46	2.370	4.738	6.7065	8.317	10.002
47	2.473	4.883	6.9094	8.671	10.114
48	2.579	4.967	6.9926	8.925	10.478
49	2.679	4.997	7.0011	8.994	10.860
50	2.770	5.000	7.0361	9.010	10.998

TABLE III—*Concluded*

k	$\theta_0 = 120°$	$\theta_0 = 125°$	$\theta_0 = 130°$	$\theta_0 = 135°$	$\theta_0 = 140°$	$\theta_0 = 145°$
1	.1655	.211	.2465	.326	.385	.447
2	.6247	.755	.8617	.945	1.007	1.056
3	.9754	1.020	1.031	1.039	1.062	1.110
4	1.015	1.054	1.166	1.342	1.541	1.729
5	1.168	1.440	1.725	1.925	2.016	2.040
6	1.624	1.916	2.014	2.024	2.091	2.267
7	1.966	2.011	2.081	2.340	2.679	2.928
8	2.009	2.165	2.578	2.919	3.019	3.036
9	2.167	2.676	2.987	3.017	3.161	3.522
10	2.623	2.993	3.031	3.339	3.802	4.010
11	2.964	3.027	3.424	3.916	4.015	4.119
12	3.006	3.362	3.932	4.014	4.266	4.780
13	3.167	3.867	4.010	4.339	4.898	5.016
14	3.623	4.006	4.280	4.915	5.016	5.320
15	3.963	4.115	4.814	5.012	5.397	5.950
16	4.005	4.600	5.006	5.338	5.961	6.022
17	4.166	4.973	5.161	5.913	6.034	6.588
18	4.623	5.013	5.721	6.011	6.537	7.008
19	4.962	5.289	6.002	6.338	6.995	7.156
20	5.008	5.811	6.076	6.913	7.079	7.831
21	5.166	6.003	6.577	7.009	7.676	8.011
22	5.623	6.082	6.981	7.338	8.006	8.383
23	5.961	6.520	7.027	7.912	8.157	8.977
24	6.004	6.947	7.424	8.009	8.798	9.054
25	6.166	7.005	7.930	8.338	9.007	9.654
26	6.623	7.222	8.007	8.912	9.265	10.008
27	6.961	7.747	8.280	9.008	9.893	10.203
28	7.004	7.998	8.843	9.338	10.011	10.875
29	7.167	8.044	9.004	9.912	10.394	11.012
30	7.624	8.440	9.160	10.008	10.958	11.450
31	7.961	8.911	9.722	10.338	11.031	11.987
32	8.003	9.003	10.000	10.911	11.537	12.080
33	8.166	9.163	10.074	11.008	11.992	12.718
34	8.623	9.676	10.577	11.338	12.077	13.008
35	8.961	9.988	10.980	11.912	12.675	13.256
36	9.003	10.023	11.025	12.007	13.003	13.911
37	9.165	10.363	11.423	12.338	13.155	14.018
38	9.623	10.865	11.929	12.911	13.797	14.518
39	9.960	11.001	12.005	13.007	14.004	14.995
40	10.003	11.113	12.280	13.337	14.263	15.111
41	10.166	11.599	12.842	13.911	14.892	15.774
42	10.623	11.970	13.003	14.006	15.009	16.004
43	10.961	12.010	13.159	14.338	15.395	16.316
44	11.003	12.288	13.722	14.911	15.957	16.942
45	11.165	12.809	13.999	15.007	16.029	17.030
46	11.624	13.000	14.074	15.338	16.537	17.586
47	11.960	13.072	14.577	15.911	16.991	18.001
48	12.003	13.519	14.979	16.006	17.076	18.152
49	12.166	13.945	15.025	16.337	17.677	18.826
50	12.624	14.003	15.424	16.911	18.002	19.005

TABLE IV

Values of the integral of Table III for all ν_k's obtained from Table II for $\theta_0 = 95°(5)130°$

k	$\theta_0 = 95°$	$\theta_0 = 100°$	$\theta_0 = 105°$	$\theta_0 = 110°$	$\theta_0 = 115°$
1	.0881	.3312	.2547	.33543	0.417
2	.2049	.4453	.4414	.51965	0.576
3	.2737	.4995	.5196	.54126	0.596
4	.3509	.5100	.5203	.55714	0.684
5	.4122	.5102	.5459	.74767	1.068
6	.4563	.5375	.6812	1.0917	1.424
7	.4837	.6197	.9289	1.3923	1.515
8	.4975	.7629	1.2076	1.5083	1.552
9	.5024	.9499	1.4134	1.5164	1.813
10	.5020	1.147	1.4991	1.6200	2.230
11	.5042	1.318	1.5076	1.9102	2.483
12	.5120	1.435	1.5395	2.2587	2.513
13	.5324	1.491	1.6796	2.4694	2.617
14	.5673	1.503	1.9289	2.5072	2.979
15	.6200	1.505	2.2070	2.5418	3.365
16	.6893	1.535	2.4118	2.7447	3.505
17	.7738	1.619	2.4964	3.0908	3.521
18	.8701	1.763	2.5047	3.3875	3.737
19	.9736	1.950	2.5381	3.5004	4.150
20	1.105	2.147	2.6790	3.5095	4.452
21	1.180	2.317	2.9289	3.6173	4.509
22	1.271	2.433	3.2068	3.9099	4.576
23	1.348	2.489	3.4106	4.2575	4.893
24	1.410	2.502	3.4946	4.4654	5.302
25	1.454	2.504	3.5037	4.5031	5.496
26	1.481	2.534	3.5374	4.5387	5.514
27	1.496	2.618	3.6789	4.7436	5.673
28	1.501	2.762	3.9288	5.0903	6.066
29	1.501	2.949	4.2066	5.3852	6.414
30	1.503	3.147	4.4102	5.4959	6.511
31	1.511	3.317	4.4947	5.5040	6.548
32	1.532	3.433	4.5028	5.6149	6.813
33	1.567	3.488	4.5369	5.9091	7.230
34	1.620	3.501	4.6784	6.2559	7.483
35	1.689	3.504	4.9289	6.4620	7.511
36	1.774	3.534	5.2066	6.4981	7.620
37	1.870	3.618	5.4102	6.5353	7.980
38	1.974	3.762	4.4936	6.7418	8.363
39	2.105	3.950	5.5017	7.0893	8.506
40	2.180	4.147	5.5367	7.3825	8.523
41	2.270	4.317	5.6783	7.4913	8.739
42	2.348	4.432	5.9290	7.5003	9.154
43	2.410	4.488	6.2065	7.6123	9.455
44	2.453	4.500	6.4100	7.9083	9.514
45	2.481	4.504	6.4936	8.2548	9.577
46	2.496	4.534	6.5017	8.4584	9.897
47	2.501	4.619	6.5361	8.4941	10.304
48	2.501	4.763	6.6800	8.5315	10.498
49	2.503	4.950	6.9291	8.7407	10.516
50	2.512	5.148	7.2065	9.0884	10.674

TABLE IV—*Concluded*

k	$\theta_0 = 120°$	$\theta_0 = 125°$	$\theta_0 = 130°$
1	.4957	.5752	.5040
2	.6184	.6920	.8404
3	.5931	.7051	.8252
4	.8862	1.141	1.146
5	1.352	1.532	1.660
6	1.531	1.568	1.689
7	1.561	1.816	2.077
8	1.881	2.336	2.557
9	2.344	2.540	2.580
10	2.516	2.601	3.028
11	2.553	3.035	3.455
12	2.880	3.468	3.483
13	3.341	3.530	3.983
14	3.512	3.736	4.349
15	3.550	4.259	4.395
16	3.878	4.518	4.939
17	4.338	4.561	5.243
18	4.509	4.950	5.318
19	4.548	5.423	5.899
20	4.878	5.520	6.124
21	5.338	5.673	6.249
22	5.507	6.183	6.837
23	5.548	6.502	7.009
24	5.879	6.537	7.175
25	6.339	6.870	7.778
26	6.508	7.375	7.894
27	6.546	7.517	8.140
28	6.878	7.623	8.709
29	7.340	8.106	8.782
30	7.508	8.483	9.096
31	7.548	8.524	9.632
32	7.878	8.796	9.673
33	8.340	9.318	10.061
34	8.508	9.515	10.545
35	8.547	9.584	10.572
36	8.877	10.026	11.017
37	9.337	10.456	11.465
38	9.509	10.518	11.478
39	9.549	10.729	11.978
40	9.877	11.253	12.346
41	10.339	11.510	12.393
42	10.509	11.554	12.936
43	10.546	11.946	13.238
44	10.880	12.420	13.318
45	11.341	12.516	13.890
46	11.508	12.671	14.125
47	11.546	13.181	14.249
48	11.880	13.500	14.839
49	12.341	13.535	15.010
50	12.509	13.869	15.176

TABLE V

Value of the integral $I_2 = \int_0^1 \{P_n(x)\}^2\,dx$ *for* $n = 0(.05)1$, *together with the second tabular difference for facilitating the interpolation n—wise of the table*

n	$\int_0^1 [P_n(x)]^2\,dx$	Δ''
0	1.00000	
05	.96833	−.00183
10	.93483	−.00154
15	.89979	−.00124
20	.86351	−.00092
25	.82631	−.00063
30	.78848	−.00032
35	.75033	−.00005
40	.71213	.00021
45	.67414	.00047
50	.63662	.00074
55	.59984	.00098
60	.56404	.00118
65	.52942	.00138
70	.49618	.00156
75	.46450	.00169
80	.43451	.00180
85	.40632	.00189
90	.38002	.00196
95	.35568	.00199
1.00	.33333	

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FORCED PERIODIC SOLUTIONS OF A STABLE NON-LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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1. We consider the system

$$dx/dt = Ax + f(x, \omega t) + kb(\omega t) \tag{1.0}$$

where (1) x is a vector with n components x_i and $|x| = \Sigma |x_i|$,

(2) A is a constant matrix the characteristic roots of which have negative real parts,

(3) the vector function $b(t)$ is continuous and has period 1 and the average of $b(t)$ over a period is zero,

(4) k and ω are constants, and

(5) the vector function $f(x, t)$ is continuous and periodic of period 1 in t and for any $\eta > 0$

$$|f(x, t) - f(y, t)| \leqq \eta |x - y| \tag{1.1}$$

for small enough $|x|$ and $|y|$ uniformly in t. Moreover $f(0, t) = 0$.

Theorem. *If k is sufficiently small or if ω is sufficiently large then* (1.0) *has a periodic solution $x = p(t)$ of period $1/\omega$. Moreover there exists a constant $\rho > 0$ such that*

$$|p(t)| \leqq \rho k/(1 + \omega).$$

The solution $p(t)$ is stable and indeed there exists an $\epsilon > 0$ such that any solution $x(t)$ of (1.0) *which at $t = t_0$ satisfies $|x(t_0)| \leqq \epsilon$ also satisfies $x(t) - p(t) \to 0$ as $t \to \infty$ providing $k/(1 + \omega)$ is small.*

An example of (1.0) is the second order differential equation

$$d^2u/dt^2 + g(u, du/dt)\, du/dt + h(u) = k \sin \omega t \tag{1.2}$$

where $g(0, 0) > 0$ and $h'(0) > 0$. A special case of (1.2) in turn is the case where g is constant and $h = u + c_1u^2 + c_2u^3$ where c_1 and c_2 are constants.

The condition (1.1) is not needed as given and can be replaced by a less restricted requirement as will be seen in the proof of the Theorem. Incidentally the successive approximations procedure used in proving the Theorem offers a practical method for getting an arbitrarily good approximation to the stable periodic solution of (1.0) when k is small or ω is large. It is the case that $b(t)$ may be piecewise continuous rather than continuous.

2. We proceed now to prove the theorem.

Proof. Part 1. Here we prove the existence of $p(t)$. Our hypothesis implies the existence of K and $\sigma > 0$ such that

$$|e^{At}| \leqq Ke^{-\sigma t}, \qquad t \geqq 0. \tag{2.0}$$

(Here the norm of a matrix $|A| = \Sigma |a_{ij}|$.) We replace (1.0) by

$$x(t) = k\int_{-\infty}^{t} e^{A(t-\tau)}b(\omega\tau)\,d\tau + \int_{-\infty}^{t} e^{A(t-\tau)}f(x(\tau),\omega\tau)\,d\tau. \tag{2.1}$$

(That is if (2.1) has a solution it follows easily that (1.0) is satisfied by the solution of (2.1).)

We set $x^{(0)}(t) = 0$ and define

$$x^{(n+1)}(t) = k\int_{-\infty}^{t} e^{A(t-\tau)}b(\omega\tau)\,d\tau + \int_{-\infty}^{t} e^{A(t-\tau)}f(x^{(n)}(\tau),\omega\tau)\,d\tau \tag{2.2}$$

Clearly

$$x^{(1)}(t) = k\int_{-\infty}^{t} e^{A(t-\tau)}b(\omega\tau)\,d\tau. \tag{2.3}$$

Let $B(t) = \int_0^t b(\tau)\,d\tau$. Then

$$x^{(1)}(t) = (k/\omega)\int_{-\infty}^{t} e^{A(t-\tau)}[dB(\omega\tau)/d\tau]\,d\tau$$

$$= (k/\omega)e^{A(t-\tau)}B(\omega\tau)]_{-\infty}^{t} + (k/\omega)\int_{-\infty}^{t} Ae^{A(t-\tau)}B(\omega\tau)\,d\tau$$

Since $|B(t)|$ is uniformly bounded we have using (2.0)

$$|x^{(1)}(t)| \leqq (kK_1/\omega) + (kK_2/\omega\sigma)$$

where K_1 and K_2 are constant. If $\omega < 1$ we have from (2.3) directly $|x^{(1)}(t)| \leqq kK_3/\sigma$ for some K_3. Thus in any case

$$|x^{(1)}(t)| \leqq kK_4/(1+\omega) \tag{2.4}$$

for some K_4.

From (2.2) we have

$$|x^{(n+1)}(t) - x^{(n)}(t)| \leqq K\int_{-\infty}^{t} e^{-\sigma(t-\tau)}\,|f(x^{(n)}(\tau),\omega\tau) - f(x^{(n-1)}(\tau),\omega\tau)|\,d\tau. \tag{2.5}$$

If $|x^{(n-1)}(t)|$ and $|x^{(n)}(t)|$ are less than some δ we have

$$|f(x^{(n)}(\tau),\omega\tau) - f(x^{(n-1)}(\tau),\omega\tau)| \leqq \tfrac{1}{2}(\sigma/K)\,|x^{(n)}(\tau) - x^{(n-1)}(\tau)|. \tag{2.6}$$

Thus if $M_n = \max|x^{(n)}(t) - x^{(n-1)}(t)|$ we have from (2.5) and (2.6)

$$M_{n+1} \leqq \tfrac{1}{2}M_n.$$

From (2.4) we have $M_1 \leqq kK_4/(1+\omega)$. Thus

$$M_n \leqq 2^{-n+1}\,kK_4/(1+\omega). \tag{2.7}$$

From (2.7) we have

$$|x^{(n)}(t)| \leqq 2kK_4/(1+\omega)$$

and therefore if we choose $k/(1+\omega)$ small enough so $2kK_4/(1+\omega) < \delta$ the above formulas are valid.

From (2.7) we see that $x^{(n)}(t)$ converges uniformly to a limit which satisfies (2.1) and which we denote by $p(t)$. The periodicity of $p(t)$ follows from that of $x^{(n)}(t)$ which in turn follows from (2.2).

Part 2. Here we prove that $p(t)$ is stable as stated in the Theorem. Let $x(t)$ be a solution of (1.0) and let $|x(0)|$ be small. (We take $t_0 = 0$ with no restriction.) We have

$$(d/dt)(x(t) - p(t)) - A(x(t) - p(t)) = f(x, \omega t) - f(p, \omega t)$$

Thus

$$x(t) - p(t) = \int_0^t e^{A(t-\tau)}[f(x(\tau), \omega\tau) - f(p(\tau), \omega\tau)]\, d\tau + e^{At}[x(0) - p(0)].$$

We can choose k small enough or ω large enough so $|p(t)| < \frac{1}{2}\delta$. For $t > 0$ and so long as $|x(t)| < \delta$ we have

$$|x(t) - p(t)| \leqq K \int_0^t e^{-\sigma(t-\tau)} \tfrac{1}{2}(\sigma/K)\,|x(\tau) - p(\tau)|\, d\tau + Ke^{-\sigma t}|x(0) - p(0)|. \tag{2.8}$$

Or

$$e^{\sigma t}|x(t) - p(t)| \leqq \tfrac{1}{2}\sigma \int_0^t e^{\sigma\tau}|x(\tau) - p(\tau)|\, d\tau + K|x(0) - p(0)|.$$

From this follows easily

$$e^{\sigma t}|x(t) - p(t)| \leqq K|x(0) - p(0)|\, e^{\sigma t/2}.$$

Or

$$|x(t) - p(t)| \leqq K|x(0) - p(0)|\, e^{-\sigma t/2}. \tag{2.9}$$

Since we can take $|x(0)|$ small and get $|p(0)|$ small by taking $k/(1+\omega)$ small we have

$$|x(t) - p(t)| < \tfrac{1}{2}\delta, \qquad t \geqq 0.$$

Since $|p(t)| < \frac{1}{2}\delta$ we have $|x(t)| < \delta$ and thus (2.8) is valid for all $t \geqq 0$ as is (2.9). From (2.9) we have $x(t) - p(t) \to 0$ as $t \to \infty$ which completes the proof of the Theorem.

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COULOMB WAVE FUNCTIONS EXPRESSED IN TERMS OF BESSEL-CLIFFORD AND BESSEL FUNCTIONS

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1. Introduction: The Computation Laboratory of the Applied Mathematics Division of the National Bureau of Standards is now preparing tables of the solution of the Coulomb Wave equation

$$d^2y/d\rho^2 + [1 - 2\eta/\rho - L(L+1)/\rho^2]y = 0. \tag{1.1}$$

This equation is of importance in the study of the interaction of particles in a Coulomb field of force. The regular solution is to be tabulated for $L = 0(1)5$, 10, 11, 20, 21, $\rho = 0(.2)5$, $\eta = -5(1)5$ together with the reduced derivatives with respect to η. The purpose of this article is to give expressions for the regular solution which will be useful outside the range of the tables. Specifically, we shall give:

(1) an expansion in terms of the modified Bessel-Clifford functions, $t^{-n/2}I_n(2\sqrt{t})$ valid when η is large [2, 3], and

(2) an expansion in terms of the Bessel functions $J_n(\rho)$.

2. Definition of the Regular Solution [1]. In applications the regular solution $y = F_L(\eta, \rho)$, of (1.1) is normalized in such a manner that as $\rho \to \infty$

$$F_L(\eta, \rho) \sim \sin(\rho - \eta \ln 2\rho - L\pi/2 + \sigma_L) \tag{2.1}$$

where $\sigma_L = \arg \Gamma(L + 1 + i\eta)$. The power series expansion of $F_L(\eta, \rho)$ is

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\phi_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\sum_{n=L+1}^{\infty} A_n^L(\eta)\rho^{n-L-1} \tag{2.2}$$

where the coefficients $A_n^L(\eta)$ are determined from the relations

$$A_{L+1}^L = 1, \qquad A_{L+2}^L = \eta/(L+1), \tag{2.3}$$

$$(n+L)(n-L-1)A_n^L = 2\eta A_{n-1}^L - A_{n-2}^L \text{ for } n > L+2.$$

The normalizing factor $C_L(\eta)$ is defined by the relations

$$C_L^2 = p_L C_0^2/2\eta(2L+1); \qquad C_0^2 = 2\pi\eta/[\exp(2\pi\eta) - 1] \tag{2.4}$$

where

$$p_L/2\eta = 1\cdot(1^2+\eta^2)(2^2+\eta^2)\cdots(L^2+\eta^2)2^{2L}/(2L+1)\,[(2L)!]^2. \tag{2.5}$$

The function $\phi_L(\eta, \rho) = u$ satisfies the differential equation

$$\rho u'' + (2L+2)u' + (\rho - 2\eta)u = 0 \tag{2.6}$$

with $u = 1$ for $\rho = 0$. We shall confine our attention to equation (2.6).

3. Expansion in terms of modified Bessel-Clifford Functions. The modified Bessel-Clifford functions are defined as

$$E_n(t) = t^{-n/2} I_n(2\sqrt{t}) = \sum_{s=0}^{\infty} \frac{t^s}{\Gamma(s+1)\Gamma(n+s+1)} \tag{3.1}$$

and they satisfy the following differential equation and recurrence relations

$$tE_n'' + (n+1)E_n' - E_n = 0, \tag{3.2}$$

$$E_n' = E_{n+1}, \tag{3.3}$$

$$tE_{n+2} = E_n - (n+1)E_{n+1}, \tag{3.4}$$

$$E_{-n} = t^n E_n. \tag{3.5}$$

With these preliminaries we can now obtain the desired solution by first making the transformation $t = 2\eta\rho$ in (2.6). We then obtain

$$tu'' + (2L+2)u' + (t/4\eta^2 - 1)u = 0. \tag{3.6}$$

Now, if $\eta \to \infty$, (3.6) becomes identical with (3.2) for $n = 2L+1$ and thus $u \to E_{2L+1}(t)$. This suggests that we try to obtain a solution of (3.6) in the form

$$u = u_0(t) + (4\eta^2)^{-1}u_1(t) + (4\eta^2)^{-2}u_2(t) + \cdots \tag{3.7}$$

Substituting (3.7) into (3.6) and equating to zero powers of $(4\eta^2)^{-1}$ we get the following system of equations

$$\begin{aligned} &tu_0'' + (2L+2)u_0' - u_0 = 0, \\ &tu_n'' + (2L+2)u_n' - u_n = tu_{n-1}; \qquad n = 1, 2, \cdots. \end{aligned} \tag{3.8}$$

This system of equations may be solved successively and we get for $n = 0, 1, 2$

$$\begin{aligned} &u_0(t) = E_{2L+1}(t), \quad u_1(t) = LE_{2L-1}(t) - \tfrac{1}{3}E_{2L-2}(t), \\ &u_2(t) = \tfrac{1}{2}L(L-1)E_{2L-3} - \tfrac{1}{15}(5L-3)E_{2L-4} + \tfrac{1}{15}E_{2L-5}. \end{aligned} \tag{3.9}$$

From the form of the expressions in (3.9) we note that the regular solution of (3.7) may be expressed in the form

$$u(t) = k(\eta)\left\{E_{2L+1}(t) + \sum_{s=1,2,\cdots}^{\infty} a_s(\eta)E_{2L-s}(t)\right\} \tag{3.10}$$

where $k(\eta)$ is a constant to be so chosen that $u(t) \equiv \varphi_L(\eta, \rho)$. Since $\varphi_L(\eta, 0) = 1$, $k(\eta)$ is determined by setting $u(0) = 1$. From (3.1) and (3.5) we have $E_{2L+1}(0) = 1/(2L+1)!$, $E_{2L-s}(0) = 1/(2L-s)!$ if $s \leq 2L$, $E_{2L-s}(0) = 0$, if $s > 2L$ and therefore

$$k^{-1}(\eta) = \frac{1}{(2L+1)!} + \sum_{s=1,2,\cdots}^{L} \frac{a_s(\eta)}{(2L-s)!}. \tag{3.11}$$

Substituting (3.10) into (3.6) we get

$$tE_{2L+1} + \sum_{s=1,2,\cdots}^{\infty} a_s(\eta)\{4\eta^2(s+1)E_{2L-s}' + tE_{2L-s}\} = 0. \tag{3.12}$$

With the aid of (3.3) and (3.4) this may be written in the form

$$E_{2L-1} - 2LE_{2L} + \sum_{s=1,2,\cdots}^{\infty} a_s(\eta)\{4\eta^2(s+1)E_{2L-s+1} + E_{2L-s-2} - (2L-s-1)E_{2L-s-1}\} = 0. \tag{3.13}$$

Then equating coefficients of E_{2L-s} to zero we get

$$\begin{aligned} &4\eta^2 a_1 = L, \quad 3(4\eta^2)a_2 = -1, \quad (4\eta^2)^2 a_3 = L(L-1)/2, \\ &4\eta^2(s+1)a_s = (2L+1-s)a_{s-2} - a_{s-3}, \quad s = 4, 5, \cdots. \end{aligned} \tag{3.14}$$

Summarizing the results of (3.10), (3.11), (3.14) we have

THEOREM

$$\varphi_L(\eta, \rho) = k(\eta)\left[E_{2L+1}(2\eta\rho) + \sum_{s=1,2,\cdots}^{\infty} a_s(\eta)E_{2L-s}(2\eta\rho)\right] \tag{3.15}$$

where

$$k^{-1}(\eta) = \frac{1}{(2L+1)!} + \sum_{s=1,2,\cdots}^{L} \frac{a_s(\eta)}{(2L-s)!}$$

and the coefficients $a_s(\eta)$ are given by the recurrence relations.

$$\begin{aligned} &4\eta^2 a_1 = L; \quad 3(4\eta^2)a_2 = -1; \quad (4\eta^2)^2 a_3 = L(L-1)/2 \\ &4\eta^2(s+1)a_s = (2L+1-s)a_{s-2} - a_{s-3} \text{ for } s = 4, 5, \cdots. \end{aligned}$$

We shall not enter into any discussion of the convergence of (3.15) but it is clear from (3.15) that the expansion will converge rapidly when η is large while $2\eta\rho$ remains fixed.

Example. To compute $\phi_L(\eta, \rho)$ for $L = 1$, $\eta = 50$, $\rho = .25$. For the particular case $t = 2\eta\rho = 25$ and

$$\phi_1(\eta, \rho) = k(\eta)\{E_3(t) + a_1E_1(t) + a_2E_0(t) + a_3tE_1(t) + \cdots\}$$

where

$$\begin{aligned} a_1 &= 1.0 \times 10^{-4} \\ a_2 &= -3.\dot{3} \times 10^{-5} \\ a_3 &= 0 \\ a_4 &= -1.\dot{3} \times 10^{-9} \\ a_5 &= 5.\dot{5} \times 10^{-10} \end{aligned}$$

$$\begin{aligned} E_3(t) &= 14.067046 \\ a_1E_1(t) &= .053420 \\ a_2E_0(t) &= -.093857 \\ a_3tE_1(t) &= .000000 \\ a_4t^2E_2(t) &= -.000076 \\ a_5t^3E_3(t) &= .000122 \\ \text{Total} &= 14.026655 \end{aligned}$$

and

$$\begin{aligned} k^{-1}(\eta) &= E_3(0) + a_1E_1(0) + a_2E_0(0) \\ &= (1/3!) + (a_1/1!) + a_2 = 5002 \div 30{,}000 \end{aligned}$$

Thus, $\phi_1(50, .25) = \dfrac{30{,}000}{5002} \times 14.026655 = 84.126279$ as compared to 84.126275

obtained with the power series (2.2). However, to achieve this accuracy with the power series seventeen terms are required. Taking only the first term in our present approximation we get 84.33 with an error of less than 1/4%. To obtain comparable accuracy with the power series ten terms would be necessary.

4. Expansion in terms of the Bessel Function $J_n(\rho)$. We shall now obtain an expansion for $\phi_L(\eta, \rho)$ in terms of the Bessel functions $J_n(\rho)$ which is useful when η is small and ρ is large. Let us assume the representation

$$\phi_L(\eta, \rho) = \sum_{n=0}^{\infty}{}' b_n(\eta) J_n(\rho); \qquad b_0 = 1 \tag{4.1}$$

Substituting in (2.6) we get

$$\sum_{n=0}^{\infty} b_n(\eta)\{\rho J_n'' + (2L+2)J_n' + \rho J_n - 2\eta J_n\} = 0 \tag{4.2}$$

From the differential equation and recurrence relations satisfied by $J_n(\rho)$, namely

$$\rho J_n'' + J_n' + \rho J_n = \frac{n^2}{\rho} J_n = \frac{n}{2}\{J_{n+1} + J_{n-1}\}; \quad 2J_n' = J_{n-1} - J_{n+1} \tag{4.3}$$

The expression (4.2) may be written in the form

$$\sum_{n=1}^{\infty} b_n(\eta)\{\tfrac{1}{2}(n+2L+1)J_{n-1} + \tfrac{1}{2}(n-2L-1)J_{n+1} - 2\eta J_n\} - 2\eta b_0 J_0 - (2L+1)b_0 J_1 = 0 \tag{4.4}$$

Then, equating coefficients of J_n to zero, we get

$$\begin{aligned} &b_0 = 1; \quad b_1 = 2\eta/(L+1); \quad (2L+3)b_2 = 8\eta^2/(L+1) + 2(2L+1) \\ &(n+2L+1)b_n = 4\eta b_{n-1} + (2L+3-n)b_{n-2}; \qquad n = 3, 4, 5. \cdots \end{aligned} \tag{4.5}$$

The convergence of (4.1) follows from the known absolute convergence of (2.2)[4].

Example. To compute $\phi_L(\eta, \rho)$ for $L = 4$, $\eta = 1$, $\rho = 5$: From (4.1) and (4.5) we have

n	b_n	$b_n J_n(5)$
0	1.0000 00	−.1775 97
1	.4	−.1310 32
2	1.7818 18	.0829 71
3	.8606 06	.3139 76
4	1.2242 42	.4789 63
5	.7186 15	.1876 59
6	.5997 11	.0785 91
7	.3295 82	.0175 92
8	.1833 80	.0033 75
9	.0773 71	.0004 27
10	.0259 40	.0000 38
11	.0051 88	.0000 02
		$\phi_4(1, 5) = .8549\ 65$

The value of $\phi_4(1, 5)$, correct to six decimal places, obtained with the power series is .8549 66. However, to achieve this accuracy, twenty-one terms of the power series are needed.

The expansion (4.1) may also be obtained with the aid of the Laplace transform. For the sake of simplicity we demonstrate the method for the case $L = 0$. If p is a complex variable with positive real part, then the Laplace transform is defined as

$$\mathfrak{L}\{\phi_0(\rho)\} = \int_0^\infty e^{-p\rho}\phi_0(\rho)\, d\rho = g(p) \tag{4.6}$$

To derive the transform of the function $\phi_0(\eta, \rho)$ we need the following properties of the Laplace transform, namely

$$\begin{aligned} \mathfrak{L}\{\phi_0'(\rho)\} &= pg(p) - \phi_0(0) \\ \mathfrak{L}\{\rho\phi_0''(\rho)\} &= -(d/dp)(p^2 g) + \phi_0(0) \\ \mathfrak{L}\{\rho\phi_0(\rho)\} &= -g'(p) \end{aligned} \tag{4.7}$$

Now, if we multiply equation (2.6) by $e^{-p\rho}$ and integrate from 0 to ∞ we obtain as a result of (4.6) and (4.7)

$$(1 + p^2)g' + 2\eta g = -1 \tag{4.8}$$

This equation is readily solved and we get

$$g(p) = \mathfrak{L}\{\phi_0(\rho)\} = k \exp\{-2\eta \arctan p\} - 1/2\eta \tag{4.9}$$

where k is the integration constant chosen so that $g(\infty) = 0$, i.e., $k = e^{\pi\eta}/2\eta$. From the well-known integral

$$\int_0^\infty e^{-pt} J_n(t)\, dt = \frac{[\sqrt{1 + p^2} - p]^n}{\sqrt{1 + p^2}} \tag{4.10}$$

one has the transform of the function $J_n(\rho)$. If we therefore assume the expansion (4.1) in the case $L = 0$, we obtain with the aid of (4.9) and (4.10)

$$\frac{\exp[\pi\eta - 2\eta \arctan p]}{2\eta} - \frac{1}{2\eta} = (1 + p^2)^{-\frac{1}{2}} \sum_{n=0}^\infty b_n(\eta)[\sqrt{1 + p^2} - p]^n \tag{4.11}$$

To determine the coefficients $b_n(\eta)$ we first put $z = \sqrt{1 + p^2} - p$ whence $p = -\frac{1}{2}(z - z^{-1})$, $\sqrt{1 + p^2} = \frac{1}{2}(z + z^{-1})$ and (4.11) becomes

$$\frac{\exp[\pi\eta + 2\eta \arctan \frac{1}{2}(z - z^{-1})]}{2\eta} = \frac{1}{2\eta} + \frac{2z}{1 + z^2} \sum_{n=0}^\infty b_n(\eta) z^n \tag{4.12}$$

If we designate the left member of (4.12) by $f(z)$, we then obtain by differentiation

$$(1 + z^2)f' = 4\eta f \tag{4.13}$$

Then substituting the right member of (4.12) in this last equation and equating powers of z we obtain the recurrence relations (4.5) for the coefficients $b_n(\eta)$ when $L = 0$.

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